

The one-dimensional Keller-Segel model with fractional diffusion of cells

Nikolaos Bournaveas*, Vincent Calvez†‡

February 11, 2010

Abstract

We investigate the one-dimensional Keller-Segel model where the diffusion is replaced by a non-local operator, namely the fractional diffusion with exponent $0 < \alpha \leq 2$. We prove some features related to the classical two-dimensional Keller-Segel system: blow-up may or may not occur depending on the initial data. More precisely a singularity appears in finite time when $\alpha < 1$ and the initial configuration of cells is sufficiently concentrated. On the other hand, global existence holds true for $\alpha \leq 1$ if the initial density is small enough in the sense of the $L^{1/\alpha}$ norm.

Keywords. Self-organization, chemotaxis, fractional diffusion, global existence, blow-up.

1 Introduction

Chemotaxis is the directed motion of cells in response to various chemical clues. It plays a key role in developmental biology, and more generally in self-organization of cell populations. Several categories of mathematical models have been proposed to describe this organization process. Depending upon the level of description required, micro-, meso- or macroscopic models can be used [27, 9, 28]. Mesoscopic models consist of kinetic (scattering) equations well-suited for describing the motion of bacteria such as *Escherichia coli* which undergo a run and tumble process [11, 5]. Macroscopic models consist of parabolic (drift-diffusion) equations and are well-suited for describing motion of large cells such as the slime mold amoebae *Dictyostelium discoideum* [17, 12, 15]. We focus on the macroscopic setting in this paper.

*University of Edinburgh, School of Mathematics, JCMB, King's Buildings, Edinburgh EH9 3JZ, UK. E-mail address: n.bournaveas@ed.ac.uk

†Unité de Mathématiques Pures et Appliquées, École Normale Supérieure de Lyon, 46 allée d'Italie 69007 Lyon, France. E-mail address: vincent.calvez@ens-lyon.fr

‡Corresponding author

The so-called Keller-Segel model exhibits a very rich behaviour, emphasized by the critical mass phenomenon arising in two dimensions of space. The Keller-Segel system writes in a simple formulation [18]:

$$\partial_t \rho(t, x) = \Delta \rho(t, x) - \nabla \cdot (\rho(t, x) \nabla c(t, x)), \quad t > 0, \quad x \in \mathbb{R}^d \quad (1.1a)$$

$$-\Delta c(t, x) = \rho(t, x). \quad (1.1b)$$

Here $\rho(t, x)$ denotes the cell density and $c(t, x)$ denotes the concentration of the chemical attractant. The first contribution in the right hand side of (1.1a) expresses the tendency of cells to diffuse under their own Brownian motion whereas the second term expresses their tendency to aggregate due to the presence of the chemical. In two dimensions of space the two opposite tendencies are evenly balanced and the global behavior of the solution depends on the total mass of cells: $M = \int_{\mathbb{R}} \rho_0(x) dx$. More precisely, for $M > 8\pi$ blow-up occurs in finite time (aggregation overwhelms diffusion) and for $M < 8\pi$ solutions are global in time (diffusion wins the competition) [4].

However in one dimension of space diffusion is always stronger than aggregation and blow-up never occurs for systems such as (1.1) [25, 16, 26].

In this paper we study the system (1.1) in one space dimension with the cell diffusion being ruled by fractional diffusion. The usual Laplacian in (1.1a) is therefore replaced by the fractional Laplacian. The non-local parabolic equation writes as follows:

$$\partial_t \rho(t, x) = -\Lambda^\alpha \rho(t, x) - \partial_x (\rho \partial_x c), \quad t > 0, \quad x \in \mathbb{R} \quad (1.2a)$$

$$-\partial_{xx}^2 c(t, x) = \rho(t, x), \quad (1.2b)$$

equipped with suitable initial condition $\rho(0, \cdot) = \rho_0$ and decay conditions at infinity. For an exponent $\alpha \in (0, 2]$, the positive operator $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ is defined in Fourier variables by $\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi)$. An alternative representation is given by:

$$\Lambda^\alpha f(x) = c_\alpha \int_{y \in \mathbb{R}} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} dy = c_\alpha \int_{h \in \mathbb{R}} \frac{2f(x) - f(x+h) - f(x-h)}{|h|^{1+\alpha}} dh,$$

where c_α is some normalizing factor.

Clearly (1.2b) does not determine c uniquely. It is customary to specify c_x , since c itself does not appear in (1.2a). Namely we opt for:

$$c_x(t, x) = -\frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(x - y) \rho(t, y) dy, \quad (1.3)$$

which is well defined for ρ having finite mass. This corresponds to the solution given by convolution with the one-dimensional Green function:

$$c(t, x) = -\frac{1}{2} \int_{\mathbb{R}} |x - y| \rho(t, y) dy, \quad (1.4)$$

which is well defined whenever the mass $\int_{\mathbb{R}} \rho(t, y) dy$ and the first moment $\int_{\mathbb{R}} |y| \rho(t, y) dy$ are finite. One can think of this solution as the limit of c_γ as

$\gamma \rightarrow 0$ where c_γ is solution to the elliptic problem: $-\partial_{xx}^2 c_\gamma + \gamma c_\gamma = \rho$. In the sequel we restrict our attention to the limiting case $\gamma = 0$ for the sake of simplicity. Global existence results would not be affected by working with $\gamma > 0$ and blow-up results would be slightly modified (due to the fast decay of the interacting kernel at infinity, see Remark 3.2).

Non-local operators, and in particular the fractional Laplacian, have received a lot of attention recently [6, 7, 8]. In biology the motivation comes from the fact that in many cases organisms adopt Lévy-flight search strategies and therefore dispersal is better modelled by non-local operators [2, 13, 14, 19, 20]. Focusing on the one-dimensional case may seem unnatural from the biological viewpoint. However we have in mind seeking a critical mass phenomenon as it has been derived for the two-dimensional classical Keller-Segel model (1.1). It appears that $\alpha = d$ is the critical fractional exponent to state such a result. Therefore it makes only sense when $d = 2$ or $d = 1$. We concentrate on the one dimensional case in this paper.

The system (1.2) was first studied in [13] where it was shown that global existence holds true for $1 < \alpha \leq 2$ assuming that $\rho_0 \in L^1 \cap L^2$ and $\rho'_0 \in L^2$. The system (1.2) has also been studied by Biler, Karch and Laurençot [3], and Li and Rodrigo [21, 22]. As opposed to [21, 22] our global existence result is more involved and our blow-up criterion is somewhat simpler. In [3] the authors do not address global existence issue, but they consider a more general setting for stating blow-up results. Both groups do not restrict to the one-dimensional case.

We aim at providing here global existence versus blow-up results in the same spirit as in the dichotomy arising in the classical two-dimensional Keller-Segel system (1.1). More precisely, we are able to prove that solutions are global in time in the 'fair-competition' case $\alpha = 1$, if the total mass M is assumed to be small enough. As far as we know, proving blow-up for large mass in the case $\alpha = 1$ is an open problem. In the case $\alpha < 1$ we show that solutions may exist globally or may blow-up depending on the initial data. We provide explicit criteria on the initial data ρ_0 which determine whether chemotactic blow-up arises or not.

In the case $\alpha > 1$ our methods can be used to improve the results of [13] by weakening the regularity hypotheses on the initial data.

Our two main results are contained in the following Theorems.

Theorem 1 (Global existence). *Consider the system (1.2) for $0 < \alpha \leq 1$ with initial data $\rho_0 \in L^{p_0}(\mathbb{R})$ for some $p_0 > 1/\alpha$. There exists a constant $K_1(\alpha)$ such that the condition,*

$$\|\rho_0\|_{L^{1/\alpha}} < K_1(\alpha), \tag{1.5}$$

guarantees existence of global weak solutions.

In addition, regularizing effects act for (1.2), and the density belongs to any L^p space for any positive time $t > 0$.

In the case $1 < \alpha \leq 2$, assume $\rho_0 \in L^{p_0}(\mathbb{R})$ for some $p_0 > 1$. Then solutions are global in time and belong to any L^p space for all positive time $t > 0$.

To complete the picture it is natural to look for blow-up results in the supercritical case. We shall prove in the sequel that the aggregation contribution can overcome the diffusion effect in the case $\alpha < 1$ under suitable restrictions on the initial data. However describing the behaviour for initial data having large mass in the case $\alpha = 1$ remains open. Our strategy fails because the constant $K_2(\alpha)$ in Theorem 2 below diverges when $\alpha \rightarrow 1$.

Theorem 2 (Blow-up). *Consider the system (1.2) for $0 < \alpha < 1$ in one space dimension with initial data $\rho_0 \in L^1((1 + |x|)dx)$. Assume in addition that the density ρ_0 is even. Then there exists a constant $K_2(\alpha)$ such that the condition,*

$$\left(\int_{\mathbb{R}} |x| \rho_0(x) dx \right)^{1-\alpha} < K_2(\alpha) M^{2-\alpha}, \quad (1.6)$$

excludes global existence of regular solution: a singularity must appear in finite time.

Remark 1.1 ($L^{1/\alpha}$ is the natural space with respect to homogeneity). *A simple scaling argument shows that $1/\alpha$ is the only exponent for which smallness assumptions (1.5) and (1.6) are admissible. As a matter of fact, one easily checks that the system (1.2) is invariant under the time-space dilation: $\rho_\lambda(t, x) = \lambda^{-\alpha} \rho(\lambda^{-\alpha} t, \lambda^{-1} x)$, $c_\lambda(t, x) = \lambda^{2-\alpha} c(\lambda^{-\alpha} t, \lambda^{-1} x)$. Furthermore this transformation preserves both the $L^{1/\alpha}$ norm and the quantity $M^{\alpha-2} \left(\int_{\mathbb{R}} |x| \rho_0(x) dx \right)^{1-\alpha}$.*

The paper is organized as follows: in Section 2 we prove global existence, beginning with a simple but not complete argument based on L^2 estimates. The proof is then achieved thanks to careful L^p estimates. The key step consists in performing suitable integration by parts with fractional diffusion. In Section 3 we prove blow-up of solutions. The paper is supplemented by numerical illustrations of the two above-mentioned phenomena.

Acknowledgements. NB would like to thank the Laboratoire Jacques-Louis Lions of the Université Pierre et Marie Curie (Paris, France) and the Département de Mathématiques et Applications of the École Normale Supérieure (Paris, France) for their hospitality and financial support during his sabbatical leave in the spring semester of 2008, during which part of the research for this paper was done. VC thanks the Centre de Recerca Matemàtica (Bellaterra, Spain), for providing an excellent atmosphere of research during the research program 'Mathematical Biology'. The authors thank anonymous referees for their fruitful comments.

2 Global existence for small initial data: proof of Theorem 1

We start by stating some estimate which will be widely used throughout this section.

Proposition 2.1 (Interpolation inequality). *For any exponents $0 < \alpha \leq 1$ and $1 \leq p < +\infty$, the following Gagliardo-Nirenberg type inequality holds true:*

$$\int_{\mathbb{R}} \rho^{p+1}(x) dx \leq \mathbf{C}(p, \alpha) \left\| \rho^{p/2} \right\|_{\dot{H}^{\alpha/2}}^2 \|\rho\|_{1/\alpha}. \quad (2.1)$$

Proof. We distinguish between the cases $\alpha < 1$ and $\alpha = 1$. In the former we use first the Hölder inequality to obtain:

$$\begin{aligned} \int_{\mathbb{R}} \rho^{p+1}(x) dx &\leq \left(\int_{\mathbb{R}} \rho^{p/(1-\alpha)}(x) dx \right)^{1-\alpha} \left(\int_{\mathbb{R}} \rho^{1/\alpha}(x) dx \right)^{\alpha} \\ &\leq \mathbf{C}(p, \alpha) \left\| \rho^{p/2} \right\|_{\dot{H}^{\alpha/2}}^2 \|\rho\|_{1/\alpha}, \end{aligned}$$

where we have used the Sobolev embedding: $\dot{H}^{\alpha/2} \hookrightarrow L^{2/(1-\alpha)}$.

In the case $\alpha = 1$, we can use the following general result [24]: for any $\lambda, \mu, s, q, r, \theta \in \mathbb{R}$ satisfying the following relations:

$$1 \leq s, q \leq r \leq \infty, \quad 0 < \theta < 1, \quad \lambda > \frac{d}{s} - \frac{d}{r}, \quad \mu < \frac{d}{q} - \frac{d}{r},$$

$$\theta \left(\lambda - \frac{d}{s} + \frac{d}{r} \right) + (1 - \theta) \left(\mu - \frac{d}{q} + \frac{d}{r} \right) = 0.$$

we have,

$$\|f\|_{L^r} \leq \mathbf{C} \|f\|_{\dot{W}^{\lambda, s}}^{\theta} \|f\|_{\dot{W}^{\mu, q}}^{1-\theta}. \quad (2.2)$$

Applying that to the particular choice: $f = \rho^{p/2}$, $\lambda = 1/2$, $\mu = 0$, $s = 2$, $r = 2(p+1)/p$, $\theta = 2/r$, $q = 2/p$ yields the result.

Observe that proceeding as above, the exponent p cannot be chosen arbitrarily (the constraint $q \geq 1$ forces $p \leq 2$). However, there is a way to extend it to any $p \geq 1$ by slightly modifying the argument: using $f = \rho^{p/2}$, $\lambda = 1/2$, $\mu = 0$, $s = 2$, $r = 2(p+1)/p$, $\theta = 1/(p+1)$, $q = 2$ we get:

$$\begin{aligned} \left(\int_{\mathbb{R}} \rho^{p+1}(x) dx \right)^{p/(2(p+1))} &\leq C \left\| \rho^{p/2} \right\|_{\dot{H}^{1/2}}^{1/(p+1)} \left(\int_{\mathbb{R}} \rho^p(x) dx \right)^{p/2(p+1)} \\ &\leq C \left\| \rho^{p/2} \right\|_{\dot{H}^{1/2}}^{1/(p+1)} \left(\|\rho\|_1 \left(\int_{\mathbb{R}} \rho^{p+1}(x) dx \right)^{p-1} \right)^{1/2(p+1)}. \end{aligned}$$

Raising this inequality to the power $2(p+1)$ leads to the result. \square

2.1 *A priori* L^2 estimates

We complete here some existing results first derived by Escudero [13]. We use Gagliardo-Nirenberg type inequalities instead of the Sobolev inequality used in [13]. This allows us to study a wider range of α 's. We are concerned in this section with the global existence of the Keller-Segel system with fractional

diffusion of cells when $1/2 \leq \alpha \leq 1$, using simple harmonic analysis estimates. This will be extended below in Section 2.2 to any $0 < \alpha \leq 1$. The purpose of this section is to derive simply *a priori* estimates which guarantee global existence of solutions and to set the stage for our approach in Section 2.2. The constraints on the exponent α here are an artefact of the method: in short the interpolation of $L^{1/\alpha}$ between L^1 and L^2 yields $1/2 \leq \alpha$.

As it is now standard in such systems, we aim at deriving suitable L^p norm of the cell density [18, 10]. Due to the simple formulation of the fractional diffusion in the Fourier space variable, we opt for $p = 2$. We will relax this constraint in the next section. We have the following estimation:

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\rho(t)\|_{L^2}^2 &= \int_{\mathbb{R}} (-\Lambda^\alpha \rho(t, x) - \partial_x (\rho(t, x) \partial_x c(t, x))) \rho(t, x) dx \\ &= - \int_{\mathbb{R}} \left(\Lambda^{\alpha/2} \rho(t, x) \right)^2 dx + \frac{1}{2} \int_{\mathbb{R}} \rho^3(t, x) dx. \end{aligned}$$

We then apply the Gagliardo-Nirenberg inequality (Proposition 2.1) for $p = 2$:

$$\int_{\mathbb{R}} \rho(t, x)^3 dx \leq \mathbf{C}(2, \alpha) \left\| \Lambda^{\alpha/2} \rho(t) \right\|_{L^2}^2 \left(\int_{\mathbb{R}} \rho^{1/\alpha}(t, x) dx \right)^\alpha. \quad (2.3)$$

In the case $\alpha = 1$ we obtain the decay of the L^2 norm providing that the mass is small enough:

$$\frac{d}{dt} \frac{1}{2} \|\rho(t)\|_{L^2}^2 \leq \left(-\frac{1}{\mathbf{C}(2, 1)M} + \frac{1}{2} \right) \int_{\mathbb{R}} \rho^3(t, x) dx. \quad (2.4)$$

It follows that $\|\rho(t)\|_{L^2} \leq \|\rho_0\|_{L^2}$, as soon as $\rho_0 \in L^2$. It is also possible to conclude without assuming $\rho_0 \in L^2$, by means of regularizing effects. In fact using interpolation between L^1 and L^3 it comes out that (2.4) also implies (when the mass is small enough):

$$\frac{d}{dt} \frac{1}{2} \|\rho(t)\|_{L^2}^2 \leq \left(-\frac{1}{\mathbf{C}(2, 1)M} + \frac{1}{2} \right) M^{-1} \|\rho(t)\|_{L^2}^4.$$

Therefore $\|\rho(t)\|_2$ becomes finite in zero time. We shall come back to that later.

In the case $\alpha < 1$ the Gagliardo-Nirenberg inequality (2.3) implies that:

$$\frac{d}{dt} \frac{1}{2} \|\rho(t)\|_{L^2}^2 \leq \left(-\frac{1}{\mathbf{C}(2, \alpha) \|\rho(t)\|_{L^{1/\alpha}}} + \frac{1}{2} \right) \int_{\mathbb{R}} \rho^3(t, x) dx.$$

As opposed to the case $\alpha = 1$, the quantity $\|\rho(t)\|_{L^{1/\alpha}}$ is not conserved in time. Therefore we have to develop an alternative strategy as in [10] for the Keller-Segel in dimension $d > 2$, where the criterion for global existence involves the $L^{d/2}$ -norm. Here we simply use the fact that $L^{1/\alpha}$ can be interpolated between L^1 and L^2 if $1/2 \leq \alpha \leq 1$. As a consequence we have:

$$\frac{d}{dt} \frac{1}{2} \|\rho(t)\|_{L^2}^2 \leq \left(-\frac{1}{\mathbf{C}(2, \alpha) M^{2\alpha-1} \|\rho(t)\|_{L^2}^{2-2\alpha}} + \frac{1}{2} \right) \int_{\mathbb{R}} \rho^3(t, x) dx.$$

Thus if the quantity $M^{2\alpha-1}\|\rho_0\|_{L^2}^{2-2\alpha}$ is small enough, then $\|\rho(t)\|_{L^2}$ automatically decays for every time. We will see later that this criterion can be ameliorated, as the $L^{1/\alpha}$ (before interpolation) appears to be the critical space for this problem (analogous to $L^{d/2}$ in the classical Keller-Segel problem). To derive this improved criterion we shall understand how the L^p norms of the cell density evolve, using more refined tools for integration by parts.

Remark 2.2. *In the case $1 \leq \alpha \leq 2$, if we assume that $\rho_0 \in L^1 \cap L^2$ and $\rho'_0 \in L^2$ we can work similarly as in [13] to obtain an a-priori estimate on $\|\rho_x(t)\|_{L^2(\mathbb{R})}$. Then the Sobolev inequality gives a bound on $\|\rho(t)\|_{L^\infty}$.*

2.2 A priori L^p estimates

Following [6, 8], the one-dimensional fractional Laplacian can be interpreted as a ‘Dirichlet to Neumann problem’ on the two-dimensional half-space (with an appropriate modification when $\alpha \neq 1$). Namely it is related to the following minimization problem. Given a function $f(x)$ defined for $x \in \mathbb{R}$ (and belonging to appropriate spaces, see [6] for details) find a function $f_*(x, y)$ defined on $\mathbb{R} \times (0, \infty)$ coinciding with $f(x)$ on the boundary: $f_*(x, 0) = f(x)$, which minimizes the weighted functional,

$$J(u) = \frac{1}{2} \int_0^\infty \int_{\mathbb{R}} |\nabla u(x, y)|^2 y^{1-\alpha} dx dy. \quad (2.5)$$

When $\alpha = 1$ this is nothing but the harmonic extension of ρ to the upper half-space. The fractional Laplacian is then deduced from the normal derivative of $\rho_*(x, y)$ on the boundary $\{y = 0\}$ as described below. We will strongly use this minimization property. It is worth noticing that the minimal value for J is the $H^{\alpha/2}$ norm of the trace f :

$$\min J(u) = \|f\|_{\dot{H}^{\alpha/2}}. \quad (2.6)$$

We refer to [6] for a proof using the Fourier characterization of the $H^{\alpha/2}$ norm.

The following Proposition enables to perform integration by parts with the fractional Laplacian. It is inspired from [8, Proposition 5].

Proposition 2.3. *Assume $\rho(x)$ is regular, then the following estimate holds true:*

$$\int_{\mathbb{R}} \rho^{p-1}(x) \Lambda^\alpha \rho(x) dx \geq \frac{4(p-1)}{p^2} \left\| \rho^{p/2} \right\|_{\dot{H}^{\alpha/2}}^2.$$

Proof. For the sake of completeness, we recall the main lines of the proof of Proposition 2.3. We begin with the case $\alpha = 1$ which is somewhat simpler.

The half-Laplacian. In short, the one-dimensional half-Laplacian $\Lambda\rho$ is the normal derivative of the harmonic extension on the upper-half plane of ρ :

$$\Lambda\rho(x) = -\partial_y \rho_*(x, 0),$$

where $\begin{cases} -\Delta \rho_*(x, y) = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ \rho_*(x, 0) = \rho(x). \end{cases}$

Using this characterization, we are able to integrate by parts and to estimate the following diffusion contribution (which appears in the proof of Theorem 1 below):

$$\begin{aligned}
\int_{\mathbb{R}} \rho^{p-1}(x) \Lambda \rho(x) dx &= \int_{\mathbb{R}} \rho_*^{p-1}(x, 0) \nabla \rho_*(x, 0) \cdot \nu dx \\
&= \int_0^\infty \int_{\mathbb{R}} \nabla \rho_*^{p-1}(x, y) \cdot \nabla \rho_*(x, y) dx dy \\
&= \frac{4(p-1)}{p^2} \int_0^\infty \int_{\mathbb{R}} |\nabla \rho_*^{p/2}(x, y)|^2 dx dy \\
&\geq \frac{4(p-1)}{p^2} \int_0^\infty \int_{\mathbb{R}} |\nabla (\rho^{p/2})_*(x, y)|^2 dx dy. \quad (2.7)
\end{aligned}$$

We have used in the last step the fact that $(\rho_*)^{p/2}$ and $(\rho^{p/2})_*$ coincide on the boundary $\{y = 0\}$. To conclude we use the fact that the minimal value for J when $\alpha = 1$ is the $H^{1/2}$ norm of ρ .

The $\alpha/2$ -Laplacian. For any $0 < \alpha < 2$ the fractional Laplacian $\Lambda^\alpha \rho$ can be interpreted as follows [6]:

$$\Lambda^\alpha \rho(x) = \lim_{y \rightarrow 0} [-y^{1-\alpha} \partial_y \rho_*(x, y)],$$

where

$$\begin{cases} -\nabla \cdot (y^{1-\alpha} \nabla \rho_*) (x, y) = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ \rho_*(x, 0) = \rho(x). \end{cases}$$

In the same lines as (2.7) we are able to estimate the following diffusion contribution:

$$\begin{aligned}
\int_{\mathbb{R}} \rho^{p-1}(x) \Lambda^\alpha \rho(x) dx &= \int_{\mathbb{R}} \rho_*^{p-1}(x, 0) y^{1-\alpha} \nabla \rho_*(x, 0) \cdot \nu dx \\
&= \int_0^\infty \int_{\mathbb{R}} \nabla \rho_*^{p-1}(x, y) \cdot y^{1-\alpha} \nabla \rho_*(x, y) dx dy \\
&= \frac{4(p-1)}{p^2} \int_0^\infty \int_{\mathbb{R}} |\nabla \rho_*^{p/2}(x, y)|^2 y^{1-\alpha} dx dy \\
&\geq \frac{4(p-1)}{p^2} \int_0^\infty \int_{\mathbb{R}} |\nabla (\rho^{p/2})_*(x, y)|^2 y^{1-\alpha} dx dy \\
&\geq \frac{4(p-1)}{p^2} \left\| \rho^{p/2} \right\|_{\dot{H}^{\alpha/2}}^2. \quad (2.8)
\end{aligned}$$

□

Proof of Theorem 1. The case $1 < \alpha \leq 2$ has already been treated in [13] (see Remark 2.4 at the end of the proof), so we focus on the situation where $0 < \alpha \leq 1$ in the sequel.

$L^{1/\alpha}$ is the critical space. Following the lines of [18] and [10] we estimate the evolution of the L^p norms of the cell density:

$$\begin{aligned} \frac{d}{dt} \frac{1}{p} \|\rho(t)\|_{L^p}^p &= - \int_{\mathbb{R}} \rho^{p-1}(t, x) \Lambda^\alpha \rho(t, x) dx + \frac{p-1}{p} \int_{\mathbb{R}} \rho^{p+1}(t, x) dx \\ &\leq - \frac{4(p-1)}{p^2} \left\| \rho^{p/2}(t) \right\|_{\dot{H}^{\alpha/2}}^2 + \frac{p-1}{p} \int_{\mathbb{R}} \rho^{p+1}(t, x) dx. \end{aligned} \quad (2.9)$$

Using Proposition 2.1 we obtain:

$$\int \rho^{p+1}(t, x) dx \leq \mathbf{C}(p, \alpha) \|\rho(t)\|_{L^{1/\alpha}} \left\| \rho^{p/2}(t) \right\|_{\dot{H}^{\alpha/2}}^2, \quad (2.10)$$

therefore

$$\frac{d}{dt} \frac{1}{p} \|\rho(t)\|_{L^p}^p \leq \left(- \frac{4(p-1)}{p^2 \mathbf{C}(p, \alpha) \|\rho(t)\|_{L^{1/\alpha}}} + \frac{p-1}{p} \right) \int_{\mathbb{R}} \rho^{p+1}(t, x) dx. \quad (2.11)$$

Choosing in particular $p = 1/\alpha$ we obtain that the $L^{1/\alpha}$ norm is time-decreasing whenever $\|\rho_0\|_{L^{1/\alpha}}$ is strictly smaller than $\frac{4\alpha}{\mathbf{C}(1/\alpha, \alpha)}$.

Regularizing effects. We shall prove within the next lines that the cell density $\rho(t, \cdot)$ belongs to any L^p space for arbitrary positive time, provided that the initial L^{p_0} norm is finite for some $p_0 > 1/\alpha$. The argument follows the main lines of [18, 10].

First we shall relax the criterion on $\|\rho_0\|_{L^{1/\alpha}}$ to

$$\|\rho_0\|_{L^{1/\alpha}} < \frac{4}{p_0 \mathbf{C}(p_0, \alpha)}. \quad (2.12)$$

This ensures that the L^{p_0} -norm, which is initially finite by assumption, is decreasing in time. As a consequence, we get the following upper-bound for any truncation $k > 0$:

$$\begin{aligned} \|(\rho(t) - k)_+\|_{L^{1/\alpha}} &\leq |\{x : \rho(t, x) > k\}|^{\alpha-1/p_0} \|(\rho(t) - k)_+\|_{L^{p_0}} \\ &\leq \left(\frac{M}{k} \right)^{\alpha-1/p_0} \|\rho_0\|_{L^{p_0}}. \end{aligned} \quad (2.13)$$

Second, we extend the above strategy to the derivation of $\|(\rho(t) - k(p))_+\|_{L^p}$ for some $k(p) > 0$ to be chosen later:

$$\begin{aligned} \frac{d}{dt} \frac{1}{p} \|(\rho(t) - k(p))_+\|_{L^p}^p &\leq - \frac{4(p-1)}{p^2} \|(\rho(t) - k(p))_+^{p/2}\|_{\dot{H}^{\alpha/2}}^2 + \frac{p-1}{p} \int_{\mathbb{R}} (\rho(t, x) - k(p))_+^{p+1} dx \\ &\quad + C(k, p) \int_{\mathbb{R}} (\rho(t, x) - k(p))_+^p dx + C(k, p) \int_{\mathbb{R}} (\rho(t, x) - k(p))_+^{p-1} dx. \end{aligned}$$

The last term can be interpolated between L^1 and L^p . The nonlinear contribution of homogeneity $p + 1$ goes as previously, except that we shall ensure here that $\|(\rho(t) - k(p))_+\|_{L^{1/\alpha}}$ is strictly smaller than $4/(p\mathbf{C}(p, \alpha))$ independently of time.

Introduce the notation: $Y_p(t) = \|(\rho(t) - k(p))_+\|_p^p$. We have,

$$\frac{d}{dt}Y_p(t) \leq \left(-\frac{4(p-1)}{p^2\mathbf{C}(p, \alpha)\|(\rho(t) - k(p))_+\|_{L^{1/\alpha}}} + \frac{p-1}{p} \right) \int_{\mathbb{R}} (\rho(t, x) - k(p))_+^{p+1} dx + O(Y_p(t)) + O(1).$$

Using the following interpolation inequality:

$$Y_p(t) \leq M^{1/p} \left(\int_{\mathbb{R}} (\rho(t, x) - k(p))_+^{p+1} dx \right)^{1-1/p},$$

we obtain for $k(p)$ large enough, thanks to (2.13),

$$\frac{d}{dt}Y_p(t) \leq -\delta Y_p(t)^{p/(p-1)} + O(Y_p(t)) + O(1),$$

where δ is a positive constant, independent of time.

As a standard consequence, the following estimate holds true for any time $t > 0$ smaller than a reference time T :

$$Y_p(t) \leq C(T)t^{1-p},$$

where the constant $C(T)$ does not depend on the initial value $Y_p(0)$. These *a priori* estimates guarantee that the L^p norms of $\rho(t)$ ($p > p_0$) becomes finite for $t > 0$.

Regularization step. So far we have only presented *a priori* estimates for the L^p norms of the solution. Once these *a priori* estimates are established the proof of global existence can be performed by well known methods (see for example [10, 4] for the Keller-Segel system) which we now describe briefly. Following [8] we introduce the regularized system

$$\partial_t \rho_\varepsilon(t, x) = \varepsilon \partial_{xx}^2 \rho_\varepsilon(t, x) - \Lambda^\alpha \rho_\varepsilon(t, x) - \partial_x (\rho_\varepsilon \partial_x c_\varepsilon), \quad t > 0, \quad x \in \mathbb{R} \quad (2.14a)$$

$$-\partial_{xx}^2 c_\varepsilon(t, x) = \rho_\varepsilon(t, x), \quad (2.14b)$$

where ε is a positive regular diffusion coefficient. It can be shown that (2.14) admits global in time solutions [8]: the drift is bounded and the one-dimensional Keller-Segel admits global in time solutions [25]. The regularization procedure does not affect the above *a priori* estimates, hence we get uniform estimates with respect to ε , that are similar to (2.11). Before passing to the limit we need to use the Aubin-Lions method [1, 23, 29, 4] to gain compactness from the time integrability of the Sobolev norm $\|\rho^{p/2}(t)\|_{H^{\alpha/2}}^2$ in (2.9),(2.10). \square

Remark 2.4 (The case $1 < \alpha \leq 2$). *Our method can also deal with $1 < \alpha \leq 2$, for which global existence has already been proved in [13]. In fact it would be possible to extend accordingly Proposition 2.1 with $\alpha/2$ derivatives ($\alpha > 1$) as follows:*

$$\int_{\mathbb{R}} \rho^{p+1}(x) dx \leq \left\| \rho^{p/2} \right\|_{\dot{H}^{\alpha/2}}^{2\beta} M^{1+p(1-\beta)}, \quad \beta = \frac{p}{p + \alpha - 1}.$$

Notice that our strategy requires weaker hypotheses on the initial data (in particular regularizing effects can be proved as before).

Remark 2.5 (Intermediate asymptotics when $\alpha = 1$). *It is known that for the classical two-dimensional Keller-Segel system the cell density in space/time rescaled variables converges to a self-similar profile when mass is subcritical [4]. The proof of this fact strongly uses the energy structure. This question is open for the one-dimensional Keller-Segel system with half-diffusion under consideration here.*

Recall that when only diffusion occurs (without a chemotactic coupling), such a self-similar decay holds true. This can be seen via the following argumentation in Fourier variables.

First rescale time and space: $u(\tau, y) = (1+t)\rho(t, (1+t)y)$, where $\tau = \log(1+t)$. The new equation reads:

$$\partial_{\tau} u(\tau, y) = -\Delta u(\tau, y) + \partial_y(yu(\tau, y)).$$

This writes in Fourier variable as follows:

$$\partial_{\tau} \hat{u}(\tau, \xi) = -|\xi| \hat{u}(\tau, \xi) - \xi \partial_{\xi} \hat{u}(\tau, \xi).$$

Or, equivalently,

$$\partial_{\tau} (\hat{u}(\tau, \xi) \exp(|\xi|)) + \xi \partial_{\xi} (\hat{u}(\tau, \xi) \exp(|\xi|)) = 0.$$

As a consequence, $\hat{u}(\tau, \xi) \exp(|\xi|)$ can be integrated along the characteristics outgoing from 0, where $\hat{u}(\tau, 0) = M$. This shows that $\hat{u}(\tau, \xi) \exp(|\xi|)$ converges to M locally in frequency. Therefore, $u(\tau, y)/M$ converges to the inverse Fourier transform of $\exp(-|\xi|)$, which is nothing but the Cauchy density.

Numerical simulations clearly indicate that such a statement is expected to hold true when a chemotactic contribution is added to the diffusion equation and mass is subcritical (see Fig. 1).

3 Blow-up: proof of Theorem 2

We focus in this section on the regime $\alpha < 1$, for which blow-up may occur. We exhibit a criterion involving the mass and the first moment of the initial cell density, in the same spirit as [10].

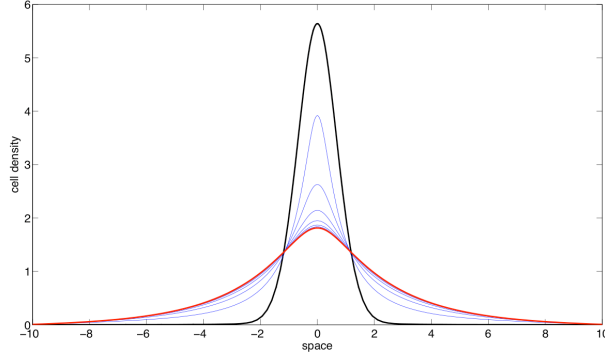


Figure 1: Numerical simulation of the one-dimensional Keller Segel system (1.2) in rescaled variables with $\alpha = 1$. The solution converges to a self-similar profile (in red). Here mass is subcritical as opposed to Fig. 2.

Testing the fractional diffusion Keller-Segel against an appropriate function ϕ (regular with constant behaviour at infinity to be precised below) writes after symmetrization:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \phi(x) \rho(t, x) dx &= \frac{c(\alpha)}{2} \iint_{\mathbb{R} \times \mathbb{R}} \frac{1}{|x-y|^{1+\alpha}} (\phi(x) - \phi(y)) (\rho(t, x) - \rho(t, y)) dx dy \\ &\quad - \frac{1}{4} \iint_{\mathbb{R} \times \mathbb{R}} \text{sgn}(x-y) (\phi'(x) - \phi'(y)) \rho(t, x) \rho(t, y) dx dy. \end{aligned} \quad (3.1)$$

We introduce a C^∞ , evenly increasing, auxiliary function ϕ satisfying: $\phi(x) = |x|$ for $|x| < 1/2$ and $\phi(x) = 1$ for $|x| > 1$. We claim that the fractional Laplacian of ϕ is a bounded function. Indeed we split the integral into two parts:

$$\begin{aligned} -\Lambda^\alpha \phi(x) &= c(\alpha) \int_{\mathbb{R}} \frac{\phi(x+h) - \phi(x)}{|h|^{1+\alpha}} dh, \\ |\Lambda^\alpha \phi(x)| &\leq \int_{|h|<2} \frac{|\phi(x+h) - \phi(x)|}{|h|^{1+\alpha}} dy + \int_{|h|>2} \frac{|\phi(x+h) - \phi(x)|}{|h|^{1+\alpha}} dh \\ &\leq \int_{|h|<2} \frac{|\phi|_{W^{1,\infty}}}{|h|^\alpha} dh + \int_{|h|>2} \frac{2}{|h|^{1+\alpha}} dh \\ &\leq |\phi|_{W^{1,\infty}} C(\alpha) + C(\alpha). \end{aligned} \quad (3.2)$$

Proof of Theorem 2. The proof begins with testing the Keller-Segel (3.1) against

the scaled function $\phi_\lambda(x) = \phi(\lambda x)/\lambda$:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \phi_\lambda(x) \rho(t, x) dx &= \int_{\mathbb{R}} (-\Lambda^\alpha \phi_\lambda(x)) \rho(t, x) dx \\ &\quad - \frac{1}{4} \int_{\mathbb{R} \times \mathbb{R}} \operatorname{sgn}(x-y) (\phi'_\lambda(x) - \phi'_\lambda(y)) \rho(t, x) \rho(t, y) dx dy. \end{aligned} \quad (3.3)$$

Thanks to a scaling argument and (3.2), we have the following estimate:

$$|\Lambda^\alpha \phi_\lambda(x)| \leq C \lambda^{\alpha-1}.$$

As a consequence we have for the first contribution in (3.3):

$$\left| \int_{\mathbb{R}} (-\Lambda^\alpha \phi_\lambda(x)) \rho(t, x) dx \right| \leq C M \lambda^{\alpha-1}.$$

On the other hand, we can write:

$$\begin{aligned} \phi(x) &= |x| + R(x), \quad R(x) = \begin{cases} 0 & \text{if } |x| < 1/2 \\ 1 - |x|, & \text{if } |x| > 1 \end{cases} \\ \phi'(x) &= \operatorname{sgn}(x) + R'(x). \end{aligned}$$

We clearly have $|R'(x)| \leq C \phi(x)$, hence:

$$|R'(\lambda x)| \leq C \phi(\lambda x) = C \lambda \phi_\lambda(x).$$

Therefore, we have for the second contribution in (3.3):

$$\begin{aligned} & - \frac{1}{4} \int_{\mathbb{R} \times \mathbb{R}} \operatorname{sgn}(x-y) (\phi'_\lambda(x) - \phi'_\lambda(y)) \rho(t, x) \rho(t, y) dx dy \\ &= - \frac{1}{4} \int_{\mathbb{R} \times \mathbb{R}} \operatorname{sgn}(x-y) (\operatorname{sgn}(\lambda x) - \operatorname{sgn}(\lambda y)) \rho(t, x) \rho(t, y) dx dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} \operatorname{sgn}(x-y) R'(\lambda x) \rho(x) \rho(y) dx dy \\ &\leq - \frac{1}{2} \int_{\{(x,y):xy<0\}} \rho(t, x) \rho(t, y) dx dy + C M \lambda \int_{\mathbb{R}} \phi_\lambda(x) \rho(t, x) dx. \end{aligned}$$

Observe that the symmetry assumption on the cell density $\rho(t, x)$ implies the crucial point:

$$\begin{aligned} \int_{\{(x,y):xy<0\}} \rho(t, x) \rho(t, y) dx dy &= 2 \left(\int_{x<0} \rho(t, x) dx \right) \left(\int_{y>0} \rho(t, y) dy \right) \\ &= \frac{M^2}{2}. \end{aligned}$$

We conclude the above estimates on the ‘corrected’ first moment $I_\lambda(t) := \int \phi_\lambda(x) \rho(t, x) dx$:

$$\begin{aligned} \frac{dI_\lambda}{dt} &\leq C M \lambda^{\alpha-1} + C \lambda^\alpha I_\lambda(t) - \frac{M^2}{4} + C \lambda M I_\lambda(t) \\ &\leq \frac{M}{4\lambda} (C \lambda^\alpha - \lambda M) + C (\lambda^\alpha + \lambda M) I_\lambda(t). \end{aligned} \quad (3.4)$$

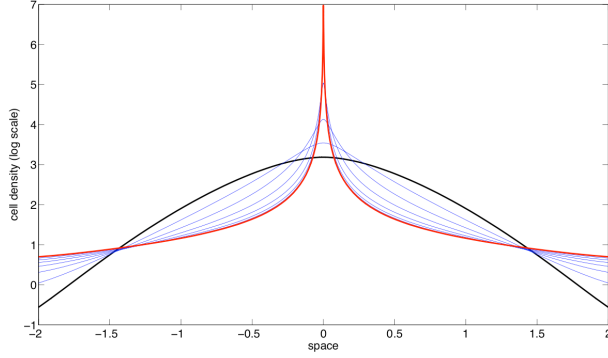


Figure 2: Numerical simulation of the one-dimensional Keller Segel system (1.2) with $\alpha = 1$ for large mass (cell density is plotted in logarithmic scale). The solution clearly blows-up (final profile is plotted in red).

We now choose λ such that the terms λ^α and λM are well-balanced, and such that $C\lambda^\alpha - \lambda M = -\lambda M/2$, which is a negative quantity. This leads to $\lambda = (\mu/M)^{1/(1-\alpha)}$, for some constant μ depending on α and the specific choice of the auxiliary function ϕ . Inequality (3.4) rewrites:

$$\frac{dI_\lambda}{dt} \leq -\frac{M^2}{8} + C\mu^{\alpha/(1-\alpha)} \frac{I_\lambda(t)}{M^{\alpha/(1-\alpha)}}.$$

To finish the argumentation, let us observe that imposing a condition of the form

$$\mu^{\alpha/(1-\alpha)} I_\lambda(0) < CM^{(2-\alpha)/(1-\alpha)}, \quad (3.5)$$

yields that the quantity I_λ must vanish in finite time, which is an obstruction to global existence.

Observe finally that $I_\lambda(0) \leq \int_{\mathbb{R}} |x| \rho_0(x) dx$, hence (3.5) is satisfied if $\int_{\mathbb{R}} |x| \rho_0(x) dx$ is sufficiently small. This completes the proof of Theorem 2. \square

Remark 3.1 (On the constants as $\alpha \nearrow 1$). *Tracking carefully the constants in the preceding proof, it turns out that μ scales like $1/(1-\alpha)$ whereas other constants are indeed of order 1. Thus criterion (3.5) rewrites:*

$$I_\lambda(0)^{1-\alpha} < C^{1-\alpha} (1-\alpha)^\alpha M^{(2-\alpha)}.$$

This clearly shows that the previous argument is not expected to be extended to the case $\alpha = 1$. However numerical simulations clearly show that a critical mass is likely to occur when $\alpha = 1$ (see Fig. 2).

Remark 3.2 (Including degradation of the chemical potential). *If we replace the Poisson equation for the chemical potential (1.4) by*

$$c_\gamma(t, x) = \int_{\mathbb{R}} B_\gamma(x - y) \rho(t, y) dy, \quad B_\gamma(x) = -\frac{1}{2} \exp(-\sqrt{\gamma}|x|), \quad (3.6)$$

we end up with the following criterion which ensures blow-up of the solution in finite time:

$$\left(\int_{\mathbb{R}} |x| \rho_0(x) dx \right)^{1-\alpha} < K_2(\alpha, \gamma) M^{2-\alpha}, \quad (3.7)$$

where $K_2(\alpha, \gamma)$ is given below (3.9). We argue as follows (omitting the index γ for the sake of clarity): the new choice for $c_x(t, x)$ induces a correction on the computation above (3.4):

$$\begin{aligned} \frac{dI_\lambda}{dt} &\leq \frac{M}{4\lambda} (C\lambda^\alpha - \lambda M) + C(\lambda^\alpha + \lambda M) I_\lambda(t) \\ &+ \frac{1}{4} \iint_{\mathbb{R} \times \mathbb{R}} \left(1 - e^{-\sqrt{\gamma}|x-y|}\right) \operatorname{sgn}(x-y) (\phi'(\lambda x) - \phi'(\lambda y)) \rho(t, x) \rho(t, y) dx dy. \end{aligned} \quad (3.8)$$

We can assume furthermore that the test function $\phi(x)$ satisfies the following properties: i) ϕ is symmetric, nondecreasing and sub-additive, and $|\phi|_{W^{1,\infty}} \leq 1$ ii) $\forall x \in \mathbb{R}, 1 - \exp(-|x|) \leq \phi(x)$. Then, the last contribution can be estimated as follows:

$$\begin{aligned} \frac{dI_\lambda}{dt} &\leq \frac{M}{4\lambda} (C\lambda^\alpha - \lambda M) + C(\lambda^\alpha + \lambda M) I_\lambda(t) \\ &\quad + \frac{1}{2} \iint_{\mathbb{R} \times \mathbb{R}} (\phi(\sqrt{\gamma}x) + \phi(\sqrt{\gamma}y)) \rho(t, x) \rho(t, y) dx dy \\ &\leq \frac{M}{4\lambda} (C\lambda^\alpha - \lambda M) + C(\lambda^\alpha + \lambda M) I_\lambda(t) + \lambda M I_\lambda(t), \quad \text{if } \sqrt{\gamma} \leq \lambda. \end{aligned}$$

As a consequence we obtain a contradiction if the first moment is initially smaller than

$$M^{(2-\alpha)/(1-\alpha)} \max_{\lambda} \left\{ \frac{1 - C\lambda^{\alpha-1}}{4\lambda(C\lambda^{\alpha-1} + C)} : \lambda \geq \sqrt{\gamma} \right\}. \quad (3.9)$$

References

- [1] J.-P. Aubin, Un théorème de compacité, *C. R. Acad. Sci. Paris* **256** (1963) 5042–5044.
- [2] F. Bartumeus, F. Peters, S. Pueyo, C. Marrasé C and J. Catalan, *Helical Lévy walks: Adjusting searching statistics to resource availability in microzooplankton*, *Proc. Natl. Acad. Sci.* **100**, 12771-5 (2003).

- [3] P. Biler, G. Karch and Ph. Laurençot, *Blowup of solutions to a diffusive aggregation model*, *Nonlinearity* **22**, 1559–1568 (2009).
- [4] A. Blanchet, J. Dolbeault and B. Perthame, *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions*, *Electron. J. Differential Equations* **44**, 32 pp. (2006).
- [5] N. Bournaveas and V. Calvez, *Global existence for the kinetic chemotaxis model without pointwise memory effects, and including internal variables*, *Kinet. Relat. Models* **1**, 29–48 (2008).
- [6] L.A. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, *Comm. Partial Differential Equations* **32**, 1245–1260 (2007).
- [7] L.A. Caffarelli and P.E. Souganidis, *Convergence of nonlocal threshold dynamics approximations to front propagation*, Preprint arXiv:0805.2618.
- [8] L.A. Caffarelli and A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Preprint arXiv:math/0608447.
- [9] F.A.C.C. Chalub, Y. Dolak-Struss, P. Markowich, D. Oelz, C. Schmeiser and A. Soreff, *Model hierarchies for cell aggregation by chemotaxis*, *Math. Models Methods Appl. Sci.* **16**, 1173–1197 (2006).
- [10] L. Corrias, B. Perthame and H. Zaag, *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, *Milan J. Math.* **72**, 1–28 (2004).
- [11] R. Erban and H.G. Othmer, *From individual to collective behavior in bacterial chemotaxis*, *SIAM J. Appl. Math.* **65**, 361–391 (2004).
- [12] R. Erban and H.G. Othmer, *Taxis equations for amoeboid cells*, *J. Math. Biol.* **54**, 847–885 (2007).
- [13] C. Escudero, *The fractional Keller-Segel model*, *Nonlinearity* **19**, 2909–2918 (2006).
- [14] C. Escudero, *Chemotactic collapse and mesenchymal morphogenesis*, *Phys. Rev. E* **72**, 022903 (2005).
- [15] T. Hillen and K.J. Painter, *A user’s guide to PDE models for chemotaxis*, *J. Math. Biol.* **58**, 183–217 (2009).
- [16] T. Hillen and A. Potapov, *The one-dimensional chemotaxis model: global existence and asymptotic profile*, *Math. Methods Appl. Sci.* **27**, 1783–1801 (2004).

- [17] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I*, Jahresber. Deutsch. Math.-Verein. **105**, 103–165 (2003).
- [18] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329**, 819–824 (1992).
- [19] J. Klafter, B.S. White and M. Levandowsky, *Microzooplankton feeding behavior and the Lévy walks*, Biological Motion, Lecture Notes in Biomathematics **89**, W. Alt and G. Hoffmann eds, Springer, Berlin, 1990.
- [20] M. Levandowsky, B.S. White and F.L. Schuster, *Random movements of soil amebas*, Acta Protozool. **36**, 237–48 (1997).
- [21] D. Li and J.L. Rodrigo, *Finite-time singularities of an aggregation equation in \mathbb{R}^n with fractional dissipation* Comm. Math. Phys. **287**, 687–703 (2009).
- [22] D. Li and J.L. Rodrigo, *Refined blowup criteria and nonsymmetric blowup of an aggregation equation*, Adv. Math. **220**, 1717–1738 (2009).
- [23] J.-L. Lions, *Équations différentielles opérationnelles et problèmes aux limites*. Die Grundlehren der mathematischen Wissenschaften, Bd. 111 Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961.
- [24] S. Machihara and T. Ozawa, *Interpolation inequalities in Besov spaces*, Proc. Amer. Math. Soc. **131**, 1553–1556 (2003).
- [25] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl. **5**, 581–601 (1995).
- [26] K. Osaki and A. Yagi, *Finite dimensional attractor for one-dimensional Keller-Segel equations*, Funkcial. Ekvac. **44**, 441–469 (2001).
- [27] B. Perthame, *PDE models for chemotactic movements: parabolic, hyperbolic and kinetic*, Appl. Math. **49**, 539–564 (2004).
- [28] B. Perthame, *Transport equations in biology. Frontiers in Mathematics*, Birkhäuser Verlag, Basel, 2007.
- [29] J. Simon, *Compact sets in the space $L^p(0, T; B)$* , Ann. Mat. Pura Appl. **146**, 65–96 (1987).