Volume effects in the Keller-Segel model: energy estimates preventing blow-up.

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Abstract

We obtain *a priori* estimates for the classical chemotaxis model of Patlak, Keller and Segel when a nonlinear diffusion or a nonlinear chemosensitivity is considered accounting for the finite size of the cells. We will show how entropy estimates give natural conditions on the nonlinearities implying the absence of blow-up for the solutions.

Key words. Chemotaxis, volume effect, prevention of overcrowding, *a priori* estimates.

Résumé

Effets de volume pour le modèle de Keller et Segel : des estimations d'énergie qui contiennent l'explosion. Nous reprenons le modèle classique de Patlak, Keller et Segel pour le chimiotactisme en considérant des termes de diffusion et de chémo-attraction non-linéaires, qui tiennent compte du fait que les cellules ont un certain volume. Nous obtenons des estimations *a priori*, et nous montrons comment de telles estimations d'entropie donnent naturellement des conditions sur ces non-linearités, pour lesquelles il n'y a pas d'explosion des solutions.

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Mots-clés. Chémotaxie, effet de volume, prévention de l'explosion, estimations *a priori*.

1 Introduction

Chemotaxis is the movement of cells oriented by chemical cues. This phenomenon occurs for a large range of cells, of different sizes and from different backgrounds. Well-known examples are the bacteria *Escherichia Coli* [Alt80], the amoeba *Dyctiostelium discoideum* [HSM95] or endothelial cells of the human body which may respond to angiogenic factors secreted by a tumor [MWO04]. Usually models for chemotaxis take into account at least two entities, namely the density of cells and the concentration of the chemical substance which is assumed to influence the movement of the population of cells.

The Patlak, Keller and Segel (PKS) model [Pat53, KS71] has been introduced in order to explain chemotactic cell aggregation by means of a coupled system of two equations: a drift-diffusion type equation for the cell density n, and a reaction-diffusion equation for the chemoattractant concentration c:

$$\begin{cases} \partial_t n - \kappa\Delta n + \nabla\cdot\Big(\chi n\nabla c\Big) = 0 \quad t\geq 0\,,\ x\in\Omega\subset\mathbb{R}^2, \\ -\Delta c = n - <n>\,. \end{cases} \tag{1.1}$$

together with the initial condition $n(0,x) = n_0(x)$ and zero-flux boundary conditions both for n and c, i.e.

$$\frac{\partial n}{\partial\eta} = \frac{\partial c}{\partial\eta} = 0 \quad t\geq 0\,,\ x\in\partial\Omega, \tag{1.2}$$

being η the outwards unit normal vector to the boundary $\partial\Omega$. Note that the system (1.1) is slightly different if the domain Ω is exactly the whole space (see [DP04] and Section 2). Parameters in this model are the diffusion coefficient κ, the chemosensitive coefficient χ, and the total mass of cells, which is formally conserved through the evolution:

$$M = \int_\Omega n\,dx.$$

It is well-known that solutions of this system may blow up in finite time (see the review paper [Hor03] and references therein). In fact there exists a threshold in the balance between the diffusion and the aggregation terms. Jäger and Luckhaus proved that there exists a constant $\mathcal{C}^*$ such that solutions are global in time whenever $\frac{\chi M}{\kappa} < \mathcal{C}^*$. They used direct *a priori* estimates based on a Gagliardo-Nirenberg-Sobolev inequality, what we will call «*Jäger & Luckhaus technique*» in the following [JL92]. It has also been shown that under an additional condition involving the second moment of n, the solution blows up in finite time if $\frac{\chi M}{\kappa} > \mathcal{C}^*_{opt}$, where

$$\mathcal{C}^*_{opt} = \begin{cases} 8\pi & \text{if } \Omega = \mathbb{R}^2, \\ 4\pi & \text{if } \Omega \text{ is a } C^2\text{, bounded, connected domain.} \end{cases}$$

Note that in the radial case the threshold for blow-up is also 8π [Nag95, Nag01, PZ04, Per05]. In the following, we will restrict ourselves to C^2, bounded, connected domains (see [GZ98] for results in the case of a piecewise C^2, bounded, connected domain).

Recently, improvements on the constant $\mathcal{C}^*$ given by Jäger and Luckhaus have been obtained both on a bounded domain by Gajewski and Zacharias [GZ98] and Biler [Bil99] (see also [BN94]), and in the whole space by Dolbeault and Perthame [DP04]. These improvements are based on fine estimates of the free energy using sharp variational inequalities. As a summary, in the linear diffusion classical PKS model one has the following result:

Theorem 1.1 (Optimal constant for Linear Diffusion PKS)**.**
Assume that $\frac{\chi M}{\kappa} < \mathcal{C}^*_{opt}$.

(i) Given a bounded initial data on a C^2*, bounded, connected domain, there exists a weak solution globally defined on time.*

(ii) Given an integrable initial data with second moment and entropy bounded, i.e., $(1 + |x|^2)n_0 \in L^1(\mathbb{R}^2)$ *and* $n_0 \log n_0 \in L^1(\mathbb{R}^2)$*, there exists a weak solution globally defined in time.*

Recent papers have pointed out the relevance of dealing with general nonlinear cell diffusion. For example Gamba *et al.* [GAC$^+$03] introduced a pressure function $\phi(n)$ in their hyperbolic model, taking into account the fact that cells do not interpenetrate, that is, they are full bodies with nonzero volume. Kowalczyk [Kow05] derived from this hyperbolic model a parabolic-elliptic system for chemotaxis where the first equation of (1.1) is replaced by

$$\partial_t n + \nabla \cdot \Big(- n \nabla h(n) + \chi n \nabla c \Big) = 0, \tag{1.3}$$

where h is related to the pressure function. Due to its biological meaning, h is an increasing function of the cell density n, which renders a saturation of the occupation number of cells. Note that the linear diffusion corresponds to $h(n) = \kappa \log n$.

On the other hand, Hillen and Painter [PH02] have included volume-filling in the model: in the context of biased random walks, they considered that the jumping probability depends on the amount of cells in the neighboring sites. Cells which are packed have less probability to move. If q denotes the correcting decreasing function, authors derive the following continuous model

$$\partial_t n + \nabla \cdot \Big(-\kappa\big(q(n) - nq'(n)\big)\nabla n + \chi_0 q(n) n \nabla c \Big) = 0. \tag{1.4}$$

In another work, assuming the chemosensitivity χ vanishes for sufficiently large cell density, they proved that the blow-up of solutions is prevented [HP01], c.f. [LW05].

The aim of our work is to present a new derivation of *a priori* estimates which lead to equi-integrability and thus, L^∞ bounds for the cell density. We do not attempt here to develop a complete existence theory for the nonlinear diffusion or chemosensitivity case. We refer to [GZ98] and [DP04] for a complete proof in the linear case.

In this paper, we improve and extend to $\mathbb{R}^2$ Kowalczyk results [Kow05] thanks to free energy methods. We show essentially that the assumption $h(u) \geq \kappa \log u$ for large u with $\kappa > \kappa^*$ and

$$\chi M = \mathcal{C}^*_{opt} \kappa^*, \tag{1.5}$$

is sufficient to prevent blow up. We will distinguish two cases: a bounded domain Ω (Section 4), and the whole space $\Omega = \mathbb{R}^2$ (Section 5). Main results are summarized in section 5.5. In Section 6 we extend our approach to the case of volume-filling type equations, i.e., a nonlinear chemosensitivity function, by connecting them to the nonlinear diffusion case, i.e., the nonlinear pressure model. In the next two sections, we will summarize the main ingredients and results of the PKS model (Section 2) and how to pass from equi-integrability bounds to L^∞ bounds (Section 3).

2 Basics of the PKS model

We first clarify the basic assumption on the diffusion coefficient.

Hypothesis H 2.1 (Basic Regularity on the Nonlinear Diffusion h)**.** *$h \in L^1_{loc}(0,\infty) \cap \mathcal{C}^1(0,\infty)$ is an increasing function with $h(1) = 0$.*

Now, we are entitled to define the functions f and Φ by: $f'(u) = uh'(u)$ with $f(0) = 0$ and $\Phi'(u) = h(u)$ with $\Phi(0) = 0$, respectively.

The nonlinear diffusion PKS model in a bounded domain $\Omega \subset \mathbb{R}^2$ consists of:

$$\left\{ \begin{array}{l} \partial_t n + \nabla \cdot \Big(- n \nabla h(n) + \chi n \nabla c \Big) = 0 \quad t \geq 0\,,\ x \in \Omega, \\ -\Delta c = n - < n > . \end{array} \right. \tag{2.1}$$

together with the initial condition $n_0 \in L^1_+(\Omega) \cap L^\infty(\Omega)$ and the no-flux boundary conditions (1.2). Note that in (2.1) the equation on c has to be understood modulo a constant, that is why we assign from now on

$$\int_\Omega c\, dx = 0, \tag{2.2}$$

when dealing with a bounded domain.

In the whole space $\mathbb{R}^2$ the equation $-\Delta c = n$ has to be understood in the sense of the Poisson kernel, and the system reads:

$$\left\{ \begin{array}{l} \partial_t n + \nabla \cdot \Big(- n \nabla h(n) + \chi n \nabla c \Big) = 0 \quad t \geq 0\,,\ x \in \mathbb{R}^2, \\ c(t,x) = -\dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \log|x-y|\, n(t,y)\, dy. \end{array} \right. \tag{2.3}$$

Hypothesis H2.2 (The initial data n_0)**.** $n_0 \in L^1_+(\Omega) \cap L^\infty(\Omega)$. *Moreover, if* $\Omega = \mathbb{R}^2$ *we assume in addition that* $n_0|x|^2 \in L^1_+(\mathbb{R}^2)$.

Remark 2.3. *These assumptions are not the optimal ones, and blow up can be prevented starting from weaker conditions (see [DP04] for instance). However we plan to give uniform bounds, and for this purpose we impose* $n_0 \in L^\infty(\Omega)$.

In the following, we will derive results valid both for the bounded domain case and for the whole space case, and thus, we will not make explicit in the integrals the domain in which we work unless it is necessary. Both systems have a common Lyapunov functional which will be crucial in the rest.

Lemma 2.4 (Free energy)**.** *Given a smooth solution of* (2.1) *and* (2.3)*, then the free energy functional [CJM+01]*

$$\mathcal{E}(t) = \int \Phi(n)\, dx - \frac{1}{2}\chi \int nc\, dx \tag{2.4}$$

verifies

$$\frac{d}{dt}\mathcal{E} = - \int n\, |\nabla(h(n) - \chi c)|^2\, dx \leq 0. \tag{2.5}$$

Example 2.5 (Power Nonlinear Diffusion). *In the case of a nonlinearity which behaves like a power : $f(u) = u^\alpha$ for some positive α, then $h(u) = \frac{\alpha}{\alpha-1}u^{\alpha-1} - \frac{\alpha}{\alpha-1}$ and $\Phi(u) = \frac{1}{\alpha-1}u^\alpha - \frac{\alpha}{\alpha-1}u$.*

We will distinguish two cases, corresponding to the two possible behaviors of h near the origin:

1. «Fast diffusion» case : $h(0^+) = -\infty$.

2. «Degenerate diffusion» case : $h(0^+) > -\infty$.

Let us recall that κ^* is defined by the critical parameter in the linear diffusion case $\chi M = \mathcal{C}^*_{opt}\kappa^*$ (1.5). The main assumption of this paper is the following:

Hypothesis H2.6 (Superlinear at ∞ Nonlinear Diffusion). *The nonlinear diffusion function h grows faster than $\kappa^* \log u$ for large u, that is, there exists $\kappa > \kappa^*$ and $\mathcal{U} \in \mathbb{R}_+$ such that*

$$\forall u \geq \mathcal{U} \quad h(u) \geq \kappa \log u.$$

Without loss of generality we assume that $\Phi(u) \geq 0$ for $u \geq \mathcal{U}$. Moreover, we assume that there exists $\delta > 0$ such that $uh'(u) \geq \delta$ for large u.

This assumption means that the behaviour of the diffusion term has to be considered only at high levels of cell density. In the following we will see that, although this hypothesis is sufficient for our purpose when dealing with a bounded domain, it has to be completed by technical assumptions in the case of the whole space. Note that the assumption on the derivative on h in (H2.6) implies the main hypothesis whenever $\delta = \kappa > \kappa^*$.

We will need the following easy consequence obtained by integrating hypothesis (H2.6) over $\{n \geq \mathcal{U}\}$.

Lemma 2.7 (Internal Energy estimate from below). *Given R such that $h(u) = \kappa \log u + R(u)$, with $R \geq 0$ for $u \geq \mathcal{U}$, then*

$$\int_{\{n \geq \mathcal{U}\}} \Phi(n)\, dx \geq \kappa \int_{\{n \geq \mathcal{U}\}} n \log n\, dx + \int_{\{n \geq \mathcal{U}\}} \mathcal{R}(n)\, dx - C(\mathcal{U}, M, \kappa), \quad (2.6)$$

where $\mathcal{R}' = R$ satisfying $\mathcal{R}(\mathcal{U}) = 0$.

Proof. We integrate the relation $h(u) = \kappa \log u + R(u)$:

$$\Phi(u) - \Phi(\mathcal{U}) = \kappa(u \log u - u - \mathcal{U} \log \mathcal{U} + \mathcal{U}) + \mathcal{R}(u).$$

Consequently

$$\int_{\{n\geq\mathcal{U}\}} \Phi(n)\,dx = \kappa \int_{\{n\geq\mathcal{U}\}} n\log n\,dx + \int_{\{n\geq\mathcal{U}\}} \mathcal{R}(n)\,dx - \kappa \int_{\{n\geq\mathcal{U}\}} C(\mathcal{U})\,dx.$$

If $C(\mathcal{U}) > 0$, we use Markov's inequality to control the last term, obtaining

$$\int_{\{n\geq\mathcal{U}\}} \Phi(n)\,dx = \kappa \int_{\{n\geq\mathcal{U}\}} n\log n\,dx + \int_{\{n\geq\mathcal{U}\}} \mathcal{R}(n)\,dx - \kappa \frac{C(\mathcal{U})}{\mathcal{U}} M.$$

and thus (2.6). □

Whereas the formulation adopted here is well-adapted to the model (1.3) involving a pressure function h, it does not match with the volume filling model (1.4). We will present in section 6 an extension in which we consider both nonlinear diffusion and chemosensitivity. In particular, we show how to reduce the analysis to a nonlinear diffusion by means of introducing a natural quantity which plays the role of the nonlinear diffusion, namely H defined by

$$H'(u) = \frac{f'(u)}{\chi(u)u}$$

and $H(1) = 0$ with a bounded chemosensitivity $\chi(u)$ positive for $u > 0$. This is obviously satisfied in the volume filling model since $\chi(u) = \chi_0 q(u)$ with q decreasing for which

$$H'(u) = \kappa \frac{q(u) - uq'(u)}{\chi_0 q(u)u}.$$

In [PH02] authors choose $q(u) = \left(1 - \frac{u}{U_{max}}\right)\mathbf{1}_{\{u<U_{max}\}}$, which leads to

$$H'(u) = \begin{cases} \dfrac{\kappa}{\chi_0}\dfrac{1}{u - U_{max}} & \text{if } u < U_{max} \\ +\infty & \text{otherwise} \end{cases}.$$

They proved in [HP01] that a vanishing effect in the chemotactic response prevents blow up in the case of linear diffusion (in our formulation it consists of setting $H'(u) = +\infty$ for large u). This is an obvious consequence of the maximum principle using the constant steady states $n = U_{max}$ in (1.4). In our case we will consider positive decreasing chemotactic coefficient $\chi(u)$ asymptotically vanishing at ∞.

Example 2.8 (Decreasing Nonlinear Chemotactic Coefficient)**.**

1. $q(u) = \frac{1}{1+u^\gamma}$, $\gamma > 0$, *leads to*

$$H'(u) = \frac{\kappa}{\chi_0} \frac{1 + (\gamma+1)u^\gamma}{u(1+u^\gamma)} \sim_\infty \frac{\kappa}{\chi_0} \frac{\gamma+1}{u},$$

   that is the diffusion corresponding to H is asymptotically linear, with the coefficient $\frac{\kappa(1+\gamma)}{\chi_0}$.

2. $q(u) = e^{-\beta u}$, $\beta > 0$, *leads to*

$$H'(u) = \frac{\kappa}{\chi_0} \frac{1+\beta u}{u} \sim_\infty \frac{\kappa}{\chi_0} \beta,$$

   that is the associated free energy functional F behaves like a square for large cell density.

3 From equi-integrability to L^∞ bound

Different approaches to get L^∞ *a priori* estimates have been proposed in the literature [JL92, Kow05]. Here, we give a sketch of the argument to derive L^∞ bounds of the cell density from equi-integrability estimates which is basically contained in the references above. In fact, the L^∞ estimate will be obtained from equi-integrability, and this considerably reduces our effort to obtain equi-integrability for both the bounded domain (Section 4) and the whole space case (Section 5). We first prove a result which shows how to gain L^p bound ($p > 2$) from equi-integrability.

The modulus of equi-integrability is denoted by

$$\omega(T, k) = \sup_{t \in [0,T]} \int (n-k)_+ \, dx.$$

Lemma 3.1 (L^p bound from equi-integrability)**.** [JL92] *Assume* (H2.1), (H2.2), (H2.6). *In addition given $T > 0$, assume the modulus of equi-integrability verifies*

$$\omega(T, k) \xrightarrow[k \to \infty]{} 0.$$

Then $n \in L^\infty\big(0, T; L^p\big)$ for $p > 2$. Moreover, if equi-integrability does not depend on time T, i.e., the previous limit is uniform in T, then $n \in L^\infty\big(\mathbb{R}_+; L^p\big)$

Proof. Let $p \geq 2$ and be k large enough so we can use the assumption (H2.6) $uh'(u) \geq \delta$, then

$$
\begin{aligned}
\frac{d}{dt}\int (n-k)_+^p\,dx \leq & -4\frac{p-1}{p}\delta\int |\nabla(n-k)_+^{p/2}|^2\,dx \\
& + \chi(p-1)\int (n-k)_+^{p+1}\,dx \\
& + p\chi k\int (n-k)_+^p\,dx + p\chi k^2\int (n-k)_+^{p-1}\,dx.
\end{aligned}
\tag{3.1}
$$

Because of the nonlinearity in the chemotactic term, we cannot apply directly a Gronwall lemma. However we can estimate the balance between the diffusion and the chemotactic contributions. Let us use the following Gagliardo-Nirenberg-Sobolev inequality [Gag59, Nir59].

$$
\int v^{p+1}\,dx \leq \mathcal{C}_{gns}(p)\int |\nabla v^{p/2}|^2\,dx \int v\,dx.
$$

We estimate the diffusion part by the chemotactic part and the modulus of equi-integrability,

$$
\begin{aligned}
\frac{d}{dt}\int (n-k)_+^p\,dx \leq & (p-1)\left(-\frac{4\delta}{pC_p\int (n-k)_+\,dx} + \chi\right)\int (n-k)_+^{p+1}\,dx \\
& + p\chi k\int (n-k)_+^p\,dx + p\chi k^2\int (n-k)_+^{p-1}\,dx.
\end{aligned}
$$

We can interpolate $(n-k)_+^p$ and $(n-k)_+^{p-1}$ between $(n-k)_+^{p+1}$ and $(n-k)_+$, and we obtain the following estimate

$$
\begin{aligned}
\frac{d}{dt}\int (n-k)_+^p\,dx \leq & (p-1)\left(-\frac{4\delta}{pC_p\,\omega(T,k)} + C(p,\chi)\right)\int (n-k)_+^{p+1}\,dx \\
& + C(p,\chi,k)\int (n-k)_+\,dx.
\end{aligned}
\tag{3.2}
$$

At this point, we choose k large enough to ensure that not only (H2.6) is satisfied but also

$$
-\frac{4\delta}{pC_p\,\omega(T,k)} + C(p,\chi) \leq -\frac{\delta}{p-1}.
$$

We can interpolate once more in (3.2), and finally we have shown that there exists $\eta > 0$ such that

$$
\frac{d}{dt}\int (n-k)_+^p \leq -\eta\int (n-k)_+^p\,dx + C(p,\chi,k,\delta)\int (n-k)_+\,dx, \tag{3.3}
$$

which guarantees that $\int (n-k)_+^p$ is bounded on $[0,T]$, and so that $n \in L^\infty\big(0,T;L^p\big)$ because of

$$\begin{aligned}\int n^p\,dx &\le \int_{\{n<k\}} k^{p-1} n\,dx + \int_{\{n\ge k\}} (n-k)^p\,dx + C(p,k)\mathrm{meas}\Big\{n \ge k\Big\}\\ &\le \int_{\{n\ge k\}} (n-k)^p\,dx + \Big(k^{p-1} + \frac{C(p,k)}{k}\Big)M.\end{aligned}$$

In addition, $\|n\|_p$ is bounded in time depending on p, χ, M, k and δ. If the equi-integrability does not depend on time, then k does not, and so the time T does not appear in the estimate giving the last assertion of this lemma $n \in L^\infty\big(\mathbb{R}_+;L^p\big)$. □

As a consequence we get immediately that n remains in L^p for some $p>2$ as soon as n is equi-integrable. We deduce from Morrey's embedding theorem that ∇c is in L^∞, and moreover that $\nabla c \in L^\infty_{loc}\big(\mathbb{R}_+;L^\infty\big)$. Thanks to the following lemma, based on an iterative method due to J. Moser [Ali79], we get that $n \in L^\infty_{loc}\big(\mathbb{R}_+;L^\infty\big)$. Let us point out that such a result was already obtained by Kowalczyk in the bounded domain case. Nevertheless, we are able to adapt his work for the whole space since all computations are led on the subset $\{n \ge k\}$ which has finite measure.

Lemma 3.2 (L^∞ bound by an iterative method)**.** [Kow05] *Assume* (H2.1), (H2.2), (H2.6)*, and also that the chemotactic potential verifies* $\nabla c \in L^\infty_{loc}\big(\mathbb{R}_+;L^\infty\big)$*, then* $n \in L^\infty_{loc}\big(\mathbb{R}_+;L^\infty\big)$ *also. Moreover, if* $\nabla c \in L^\infty\big(\mathbb{R}_+;L^\infty\big)$ *then* $n \in L^\infty\big(\mathbb{R}_+;L^\infty\big)$.

Proof. We just describe in the following the main steps of the proof and refer to [Kow05] for more details. The main idea is to use the $\|\nabla c\|_\infty$ estimate in order to decrease the homogeneity of the chemotactic term. Assume $p \ge 2$ and k large enough to ensure the applicability of hypotheses (H2.6), then we deduce a similar estimate as in the lemma 3.1, except that the right-hand side term involves $\int (n-k)_+^{p/2}\,dx$.

$$\begin{aligned}\frac{d}{dt}\int (n-k)_+^p\,dx \le &-p^2 C\|\nabla c\|_\infty^2 \int (n-k)_+^p\,dx\\ &+ C^2 p^4 \frac{\|\nabla c\|_\infty^4}{\delta^2}\left(\int (n-k)_+^{p/2} dx\right)^2\\ &+ p^2 C \|\nabla c\|_\infty^2,\end{aligned}$$

where C is a generic constant depending only on δ, χ, M and k (C does not depend on p !).

A refined study [Kow05, Lemma 5.1] of this differential inequality is sufficient to propagate bounds for

$$\int (n-k)_+^{2^j}\,dx,$$

for $j \geq 0$, and to prove L^∞ bound for n. We only highlight that $\|(n_0 - k)_+\|_p^p \leq \text{meas}\Big\{n_0 \geq k\Big\}\|n_0\|_\infty^p$. Consequently let us choose any $T > 0$ and define

$$K_j = \sup_{t\in[0,T]} \int (n-k)_+^{2^j}\,dx,$$

then

$$K_j \leq C \max\Big(\|n_0\|_\infty^{2^j}\ ,\ 2^{2j}K_{j-1}^2 + C\Big). \tag{3.4}$$

Because $a + b \leq 2\max(a, b)$ we reduce to

$$L_j \leq C \max\Big(1\ ,\ 2^{2j}L_{j-1}^2\Big),$$

where $K_j = L_j\|n_0\|_\infty^{2^j}$; furthermore we deal with the following recurrence,

$$\log_+ L_j \leq 2\log_+ L_{j-1} + j\log 4 + C.$$

Because $\sum j2^{-j}$ is convergent it ensures that $2^{-j}\log L_j$ is bounded, and finally we can pass to the limit $j \to \infty$. This proves that $n \in L^\infty\big(0, T; L^\infty(\Omega)\big)$. $\square$

Summarizing, these two previous lemmas imply that $n(t, \cdot)$ is in L^∞ whenever n is equi-integrable in the sense precised in Lemma 3.1. Moreover, the L^∞ estimate is uniform or local in time whether equi-integrability is uniform or local in time.

4 *A priori* estimates on a bounded domain

Our aim is now to prove that the cell density $n(t, \cdot)$ is equi-integrable. In the linear case it is a common way to look for estimates like $\int n \log n$ or even any functional of n growing faster than n. In this nonlinear context, Φ plays the role of this functional. First of all, if Ω is a bounded domain, we can prove directly that both terms of the energy are bounded and particularly that n is equi-integrable.

Theorem 4.1 (Equi-integrability in Ω bounded)**.** *Assume* (H2.1), (H2.2), (H2.6) *then*

$$\lim_{k\to\infty} \sup_{t\geq 0} \int_\Omega (n-k)_+ \, dx = 0,$$

and thus, $n \in L^\infty\big(\mathbb{R}_+; L^\infty(\Omega)\big)$*.*

Proof. We first rewrite the free energy as

$$\mathcal{E}(t) = \int_\Omega \big\{\Phi(n) - \chi n c\big\} \, dx + \frac{\chi}{2} \int_\Omega |\nabla c|^2 \, dx = J_c[n] + \frac{\chi}{2} \int_\Omega |\nabla c|^2 \, dx \quad (4.1)$$

and we remind that due to Lemma 2.5 it is a non increasing function of time, that is, $\mathcal{E}(t) \leq \mathcal{E}_0$.

Step 1: Explicit estimate for $\nabla c \in L^2$. Given $c \in W^{1,1}(\Omega)$, the convex functional $J_c[n]$ has a critical point n^* which is solution of

$$h(n^*) - \chi c = \lambda, \quad (4.2)$$

whenever $n^* > 0$ and null otherwise. Here, λ is the Lagrange multiplier associated to the constraint given by mass conservation $\int_\Omega n^* = M$ and fixed by this condition. We refer to [CJM+01, Proposition 5] for details. Therefore, we have

$$J_c[n] \geq \int_\Omega \big\{\Phi(n^*) - \chi n^* c\big\} \, dx = \int_{\{n^*>0\}} \big\{\Phi(n^*) - n^* h(n^*) + \lambda n^*\big\} \, dx.$$

In order to estimate precisely the right-hand side term, and particularly λ, we introduce the corrective term R such that $h(n^*) = \kappa \log n^* + R(n^*)$, then

$$J_c[n] \geq \int_\Omega \big\{\Phi(n^*) - \kappa n^* \log n^*\big\} \, dx - \int_{\{n^*>0\}} n^* R(n^*) \, dx + \lambda M. \quad (4.3)$$

Moreover, (4.2) implies $\kappa \log n^* + R(n^*) = \lambda + \chi c$ whenever $n^* > 0$ and thus

$$\int_{\{n^*>0\}} \exp\left(\frac{R(n^*)}{\kappa}\right) n^* \, dx = e^{\lambda/\kappa} \int_{\{n^*>0\}} \exp\left(\frac{\chi}{\kappa} c\right) \, dx$$

and

$$\lambda = \kappa \log\left(\int_{\{n^*>0\}} e^{R/\kappa} n^* \, dx\right) - \kappa \log\left(\int_{\{n^*>0\}} e^{\chi c/\kappa} \, dx\right). \quad (4.4)$$

If we replace λ by this expression in inequality (4.3), we conclude that

$$
\begin{aligned}
J_c[n] \geq & \int_\Omega \left\{\Phi(n^*) - \kappa n^* \log n^*\right\} dx - \int_{\{n^*>0\}} n^* R(n^*)\, dx \\
& + \kappa M \log\left(\int_{\{n^*>0\}} e^{R/\kappa} n^*\, dx\right) - \kappa M \log\left(\int_{\{n^*>0\}} e^{\chi c/\kappa}\, dx\right)
\end{aligned} \tag{4.5}
$$

On one hand, assumption (H2.6) and Lemma 2.7 tell us that

$$\int_{\{n^* \geq \mathcal{U}\}} \left\{\Phi(n^*) - \kappa n^* \log n^*\right\} dx \geq C,$$

by (2.6). On the other hand, we trivially have

$$\int_{\{n^* < \mathcal{U}\}} \left\{\Phi(n^*) - \kappa n^* \log n^*\right\} dx \geq -\Big(\sup_{[0,\mathcal{U})} \left(\Phi - \kappa n \log n\right)^-\Big) |\Omega|.$$

Therefore,

$$\int_\Omega \left\{\Phi(n^*) - \kappa n^* \log n^*\right\} dx$$

is bounded uniformly from below.

Now, the Jensen inequality for the probability density n^*/M over the set where $n^* > 0$, gives us that

$$\exp\left(\int_{n^*>0} \frac{R(n^*)}{\kappa} \frac{n^*}{M}\, dx\right) \leq \int_{n^*>0} e^{R/\kappa} \frac{n^*}{M}\, dx,$$

and thus,

$$\kappa M \log\left(\int_{n^*>0} e^{R/\kappa} \frac{n^*}{M}\, dx\right) - \int_{n^*>0} n^* R(n^*)\, dx \geq 0.$$

Finally, let us use the Trudinger-Moser inequality:

Theorem 4.2 (Trudinger-Moser inequality)**.** [Mos71, CY88, GZ98] *Suppose that $\Omega \subset \mathbb{R}^2$ is a C^2, bounded, connected domain. It exists a constant $\mathcal{C}_\Omega$ such that for all $h \in H^1$ with $\int_\Omega h = 0$ we have*

$$\int_\Omega \exp(|h|)\, dx \leq \mathcal{C}_\Omega \exp\Big(\frac{1}{8\pi} \int_\Omega |\nabla h|^2\, dx\Big).$$

applied to $\chi c/\kappa$ to conclude

$$\int_{n^*>0} e^{\chi c/\kappa}\, dx \le \int_\Omega e^{\chi c/\kappa}\, dx \le C \exp\left(\frac{\chi^2}{8\pi\kappa^2}\int_\Omega |\nabla c|^2\, dx\right),$$

and thus,

$$-\kappa M \log\left(\int_{n^*>0} e^{\chi c/\kappa}\, dx\right) \ge -\frac{\chi^2}{8\pi\kappa} M \int_\Omega |\nabla c|^2\, dx.$$

Consequently, we have quite precisely estimated the free energy (2.4) in the case of a bounded domain, and if C denotes a generic constant, combining (4.1) and (4.5) we get

$$\mathcal{E}_0 \ge \frac{\chi}{2}\left(1 - \frac{\chi M}{4\pi\kappa}\right)\int_\Omega |\nabla c|^2\, dx + C. \tag{4.6}$$

Finally, assumption (H2.6) implies that $\kappa > \kappa^*$, *i.e.*, $1 - \frac{\chi M}{4\pi\kappa} > 0$ and thus

$$\int_\Omega |\nabla c|^2\, dx$$

is uniformly bounded.

Step 2: Equi-integrability of n. Because we have started from

$$\mathcal{E}_0 \ge \mathcal{E}(t) = \int_\Omega \Phi(n)\, dx - \chi \int_\Omega |\nabla c|^2\, dx,$$

we get from (4.6) that

$$\int_\Omega \Phi(n)\, dx$$

is also uniformly bounded. In addition, assumptions (H2.1) and (H2.6) implies that Φ is a continuous bounded from below function positive outside an interval $[0, \mathcal{U}]$, and thus

$$\int_\Omega \Phi^-(n)\, dx = \int_{0\le n\le\mathcal{U}} \Phi^-(n)\, dx \ge -\big(\sup \Phi^-\big)|\Omega|.$$

Therefore we are ensured that

$$\int_\Omega \Phi^+(n)\, dx$$

is uniformly bounded in time being the function $\Phi^+(u)$ superlinear at infinity due again to assumption (H2.6). This condition is classically known to be sufficient for equi-integrability, i.e.,

$$\lim_{k\to\infty} \sup_{t\geq 0} \int_\Omega (n-k)_+ \, dx = 0.$$

Step 3: propagation of L^p bounds. Applying Lemmas 3.1 and 3.2 of Section 3, we know that not only $n \in L^\infty\big(\mathbb{R}_+; L^p(\Omega)\big)$ for all $1 \leq p < \infty$, but also $n \in L^\infty\big(\mathbb{R}_+; L^\infty(\Omega)\big)$. □

5 *A priori* estimates in the whole space

In the whole space the model analysis is more complicated because we require some control of the cell density n for large values of $|x|$. We are looking for such additional information both to justify that we can pass to the limit in the approximation phase for existence; and to estimate $\int \Phi^-(n)$ when it is necessary (in the standard PKS model $\Phi(u) = u \log u$) since n will decay somehow for large values of $|x|$. For this purpose the second moment of $n(t,\cdot)$, i.e.,

$$I\!I(t) = \frac{1}{2} \int_{\mathbf{R}^2} |x|^2 n(t,x) \, dx$$

will be our key quantity – see [DP04] and [CPZ04] for details in the linear PKS model.

Therefore, we need to distinguish several cases depending on the behavior of the diffusion for small values of the density n or large values of $|x|$.

5.1 Equi-integrability: Degenerate diffusion

Let us first assume in this section that we deal with degenerate diffusion.

Hypothesis H 5.1. *We assume that* $h(0^+) > -\infty$.

In the whole space case, the free energy (2.4)

$$\mathcal{E}(t) = \int_{\mathbb{R}^2} \Phi(n) \, dx - \frac{\chi}{2} \int_{\mathbb{R}^2} nc \, dx$$

can be rewritten as

$$\mathcal{E}(t) = \int_{\mathbb{R}^2} \Phi(n) \, dx + \frac{\chi}{4\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y| n(x) n(y) \, dx \, dy. \tag{5.1}$$

The second term in the right-hand side is well adapted to the Hardy-Littlewood-Sobolev inequality:

Theorem 5.2 (The logarithmic Hardy-Littlewood-Sobolev inequality)**.** [CL92, DP04] *Assume f is a nonnegative function $\mathbb{R}^2 \to \mathbb{R}$ with total mass M and $f(x)\log(1+|x|^2)$ integrable, then*

$$-\int_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\log|x-y| f(y)\,dx\,dy \leq \frac{M}{2}\int_{\mathbb{R}^2} f\log f\,dx + C$$

where $C = M^2(1+\log\pi+\log M)/2$ is optimal.

We deduce from this sharp estimate that

$$\mathcal{E}_0 \geq \mathcal{E}(t) \geq \int_{\mathbb{R}^2} \Phi(n)\,dx - \frac{\chi M}{8\pi}\int n\log n\,dx + C. \tag{5.2}$$

The assumption of degenerate diffusion (H5.1) is useful to control Φ near 0, since it implies that $\Phi(u) \geq Au$ with $A = h(0^+)$ for all $u \geq 0$. Now, denoting by Θ the functional

$$\Theta(u) = \Phi(u) - Au - \kappa^* u\log u, \tag{5.3}$$

it is clear from (H2.6) and Lemma 2.7 that Θ is growing faster than linearly, that is

$$\lim_{u\to\infty}\frac{\Theta(u)}{u} = +\infty,$$

and that Θ is positive for large $u \geq r > 0$. Moreover, by (H5.1), $\Theta(u) \geq -\kappa^* u\log u$, and thus,

$$\int_{\mathbb{R}^2}\Theta^-(n)\,dx = \int_{\{1\leq n\leq r\}}\Theta^-(n)\,dx \leq \int_{\{1\leq n\leq r\}}\kappa^* u\log u\,dx \leq \kappa^* M\log r. \tag{5.4}$$

Combining (5.2) and (5.4), we get consequently the estimate

$$\int_{\mathbb{R}^2}\Theta^+(n)\,dx \leq C, \tag{5.5}$$

where C does not depend on time and $\Theta^+(u)$ is growing faster than linearly. We deduce as previously the following statement.

Theorem 5.3 (Equi-integrability for degenerate diffusion)**.** *Assume* (H2.1), (H2.2), (H2.6) *and* (H5.1)*, then*

$$\lim_{k\to\infty}\sup_{t\geq 0}\int (n-k)_+\,dx = 0,$$

and therefore, $n \in L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^2))$.

5.2 Equi-integrability: a non optimal constant – The Jäger & Luckhaus' technique

Here we prove that if $uh'(u) \geq \mathcal{H}$ for large u, and sufficiently large $\mathcal{H}$ then we get equi-integrability for n without any time dependance in the bounds.

Hypothesis H 5.4 (Kowalczyk)**.** *There exists A such that $uh'(u) \geq \mathcal{H}$ for $u \geq A$; where $\mathcal{H} > \frac{3}{4}\chi M \mathcal{C}_{gns}$.*

Here, $\mathcal{C}_{gns}$ refers to the optimal constant in a Gagliardo-Nirenberg-Sobolev inequality used below.

Theorem 5.5 (Equi-integrability for a non optimal constant)**.** *Assume* (H2.1), (H2.2), (H2.6) *and* (H5.4)*, then*

$$\lim_{k\to\infty} \sup_{t\geq 0} \int_{\mathbb{R}^2} (n-k)_+ \, dx = 0.$$

Therefore $n \in L^\infty\big(\mathbb{R}_+; L^\infty(\mathbb{R}^2)\big)$.

Proof. Here, we reproduce the Jäger and Luckhaus' [JL92] arguments:

$$\begin{aligned}
\frac{d}{dt}\int_{\{n\geq A\}} (n-A)^2 \, dx = &- 2\int_{\{n\geq A\}} \nabla(n-A)\cdot\nabla f(n)\, dx \\
&+ 2\chi \int_{\{n\geq A\}} \nabla(n-A)\cdot n\nabla c \, dx \\
= &- 2\int_{\{n\geq A\}} nh'(n)|\nabla(n-A)|^2 \, dx \\
&+ \chi \int_{\{n\geq A\}} \big\{(n-A)^2 + 2A(n-A)\big\} n \, dx,
\end{aligned}$$

and thus,

$$\begin{aligned}
\frac{d}{dt}\int_{\{n\geq A\}} (n-A)^2 \, dx \leq &- 2\mathcal{H}\int_{\{n\geq A\}} |\nabla(n-A)|^2 \, dx \\
&+\chi\int_{\{n\geq A\}} \big\{(n-A)^3 + 3A(n-A)^2 + 2A^2(n-A)\big\}\, dx.
\end{aligned} \tag{5.6}$$

We can easily bound the polynomial in the last integral using $2A(n-A)^2 \leq (n-A)^3 + A^2(n-A)$ and we apply the following Gagliardo-Nirenberg-Sobolev inequality [Gag59, Nir59]

$$\int_{\mathbb{R}^2} w^3 \, dx \leq \mathcal{C}_{gns} \int_{\mathbb{R}^2} |\nabla w|^2 \, dx \int_{\mathbb{R}^2} w \, dx,$$

to conclude that

$$\frac{d}{dt}\int_{\mathbb{R}^2}(n-A)_+^2 dx \leq \left(-\frac{2\mathcal{H}}{M\mathcal{C}_{gns}}+\frac{3}{2}\chi\right)\int_{\mathbb{R}^2}(n-A)_+^3\,dx+\frac{7}{2}\chi A^2 M. \quad (5.7)$$

If $\mathcal{H}$ is chosen sufficiently large so that $\eta = \frac{2\mathcal{H}}{M\mathcal{C}_{gns}} - \frac{3}{2}\chi > 0$ then we get immediately equi-integrability uniformly in time. Indeed $2(n-A)^2 \leq (n-A)^3 + (n-A)$, and a consequence of (5.7) is

$$\frac{d}{dt}\int_{\mathbb{R}^2}(n-A)_+^2\,dx \leq -2\eta\int_{\mathbb{R}^2}(n-A)_+^2\,dx+\eta M+\frac{7}{2}\chi A^2 M. \quad (5.8)$$

from which the theorem concludes. □

Remark 5.6. *The choice of the functional $\Psi(u) = (u-A)_+^2$ growing faster than linearly is almost arbitrary. Of course, another functional will lead to another constant, but our aim in this section is definitely not to exhibit a best constant. Here, we have shown that if we are not interested in an optimal growth of the nonlinearity, then equi-integrability is gained by assuming hypothesis* (H5.4).

5.3 Cell density control at ∞

We would like to get a control of n near infinity to avoid a potential mass loss at ∞. We plan to reproduce the computation of the second moment $I\!I(t)$ [CPZ04, Per05].

Lemma 5.7 (Avoiding loss of mass at ∞: degenerate diffusion)**.** *Assume* (H2.1), (H5.1) *and that the solution satisfies $n \in L^\infty_{loc}\big(\mathbb{R}_+; L^\infty(\mathbb{R}^2)\big)$, then $II(t) \in L^\infty_{loc}(\mathbb{R}_+)$. If the L^∞ bound on the density is uniform, $n \in L^\infty\big(\mathbb{R}_+; L^\infty(\mathbb{R}^2)\big)$, then $I\!I(t)$ increases at most linearly in time.*

Proof. By computing formally the evolution of the second moment in (2.3), we get

$$\frac{d}{dt}I\!I(t) = 2\int_{\mathbb{R}^2} f(n)\,dx - \frac{\chi}{4\pi}M^2. \quad (5.9)$$

The assumptions (H2.1) and (H5.1) ensures that $\frac{f(u)}{u}$ is bounded near zero. If the solution verifies $n \in L^\infty\big(\mathbb{R}_+; L^\infty(\mathbb{R}^2)\big)$, we deduce that

$$\frac{d}{dt}I\!I(t) \leq C\int_{\mathbb{R}^2} n\,dx = CM.$$

It is easy to conclude if $n \in L^\infty_{loc}\big(\mathbb{R}_+; L^\infty(\mathbb{R}^2)\big)$. □

Let us give an alternative hypothesis to get a substitute of Lemma 5.7 for non-degenerate diffusions. Indeed the assumption $f(u) \le Cu$ is not generally met near zero and although $n \in L^\infty\big(\mathbb{R}_+; L^\infty(\mathbb{R}^2)\big)$, it is not easy to estimate directly the contribution of

$$\int_{\mathbb{R}^2} f(n)\, dx$$

in (5.9). Let us consider $\gamma(u) = \dfrac{f(u)}{u}$.

Hypothesis H 5.8. *Given $h(0^+) = -\infty$, we assume that γ is strictly decreasing on an interval $(0, \gamma^*)$, $\gamma(0^+) = \infty$ and that $f \circ \gamma^{-1}$ is integrable near infinity.*

Remark 5.9. *In the particular case of a power behavior near zero, $f(u) = \kappa u^\alpha \quad \forall u < a$ with $\alpha < 1$, previous hypothesis (H5.8) is equivalent to $\alpha > \frac{1}{2}$. This excludes too fast diffusions near zero.*

Lemma 5.10 (Avoiding loss of mass at ∞: fast diffusion)**.** *Assume* (H2.1), (H5.8) *and that the solution verifies $n \in L^\infty_{loc}\big(\mathbb{R}_+; L^\infty(\mathbb{R}^2)\big)$, then $I\!I(t) \in L^\infty_{loc}(\mathbb{R}_+)$.*

Proof. Given $T > 0$, let us consider $U = \|n\|_{L^\infty((0,T);L^\infty(\mathbb{R}^2))}$. Now, we can fix any $0 < a < \min(\gamma^*, U)$ to estimate

$$\int_{\{a \le n \le U\}} f(n)\, dx \le \Big(\max_{u \in [0,U]} f(u)\Big) \text{meas}\Big\{n \ge a\Big\} \le \Big(\max_{u\in[0,U]} f(u)\Big)\frac{M}{a}.$$

Now, we can restrict to the set $\{n < a\}$ and split the integral as

$$\begin{aligned}\int_{\{n<a\}} f(n)\, dx &= \int_{\{n<a\} \bigcap \{\gamma(n) \le |x|^2\}} f(n)\, dx + \int_{\{n<a\} \bigcap \{\gamma(n) > |x|^2\}} f(n)\, dx \\ &\le \int_{\mathbb{R}^2} |x|^2 n\, dx + \int_{\{n<a\} \bigcap \{\gamma(n) > |x|^2\}} f(n)\, dx.\end{aligned}$$

We split again the last term into

$$\int_{\{n<a\} \bigcap \{\gamma(a) < |x|^2 < \gamma(n)\}} f(n)\, dx + \int_{\{n<a\} \bigcap \{|x|^2 \le \gamma(a)\}} f(n)\, dx. \tag{5.10}$$

The second term is easily controlled because we have reduced to a bounded domain, that is

$$\int_{\{n<a\} \bigcap \{|x|^2 \le \gamma(a)\}} f(n)\, dx \le \pi \left(\max_{u \in [0,a]} f(u)\right) \gamma(a).$$

Dealing with the first term of (5.10) we can invert γ and consequently

$$\begin{aligned}\int_{\{n<a\}\bigcap\{\gamma(a)<|x|^2<\gamma(n)\}} f(n)\,dx \leq& \int_{\{|x|^2>\gamma(a)\}} |f\circ\gamma^{-1}(|x|^2)|\,dx\\ =& \int_{\gamma(a)}^{\infty} |f\circ\gamma^{-1}(r^2)|2\pi r\,dr\\ =&\pi\int_{\sqrt{\gamma(a)}}^{\infty} |f\circ\gamma^{-1}(s)|\,ds.\end{aligned}$$

Combining previous estimates, we deduce

$$\int_{\mathbb{R}^2} f(n)\,dx \leq 2\mathit{\Pi}(t) + C_T, \tag{5.11}$$

for all $0 \leq t \leq T$, that together with (5.9) implies the stated result. $\square$

5.4 Equi-integrability: fast diffusion – An energy method

By using (H2.6), we can rewrite the energy estimate (5.2) as follows

$$\mathcal{E}_0 \geq \mathcal{E}(t) \geq \int_{\{n<\mathcal{U}\}} \Phi(n)\,dx + \int_{\{n\geq\mathcal{U}\}} \Phi(n)\,dx - \frac{\chi M}{8\pi}\int n\log n\,dx + C.$$

We split again the right-hand side term and we get thanks to (2.6)

$$\int_{\{n<\mathcal{U}\}} \Phi(n)\,dx + \int_{\{n\geq\mathcal{U}\}} \mathcal{R}(n)\,dx + (\kappa-\kappa^*)\int_{\{n\geq\mathcal{U}\}} n\log n\,dx + C \leq \mathcal{E}_0,$$

and thus,

$$\int_{\{n<\mathcal{U}\}} \Phi(n)\,dx + \Big(1-\frac{\kappa^*}{\kappa}\Big)\int_{\{n\geq\mathcal{U}\}} \Phi(n)\,dx + C \leq \mathcal{E}_0. \tag{5.12}$$

Let us recall that $\Phi \geq 0$ for $u \geq \mathcal{U}$ (H2.6). Our problem now is to estimate the potential negative contribution arising from

$$\int_{\{n<\mathcal{U}\}} \Phi(n)\,dx \tag{5.13}$$

in the fast diffusion case. Let us remind that in the degenerate diffusion case, $|\Phi(u)|$ is dominated by u near the origin giving a simple argument to control this negative contribution in section 5.1.

We propose to couple the evolution of the second moment of n into the computations, more precisely, to couple the second moment evolution and the behaviour of (5.13). In fact, we will proceed analogously to Lemma 5.10 without the assumption of boundedness of n since we will work on the set $\{n < \mathcal{U}\}$. Let us consider $\beta(u) = \dfrac{|\Phi(u)|}{u}$.

Hypothesis H 5.11. *Given* $h(0^+) = -\infty$*, we assume that* β *is strictly decreasing on an interval* $(0, \beta^*)$*,* $\beta(0^+) = \infty$ *and that* $|\Phi| \circ \beta^{-1}$ *is integrable near infinity.*

Lemma 5.12 (Control of the negative contribution of the internal energy: fast diffusion)**.** *Assume* (H2.1) *and* (H5.11)*, then*

$$\int_{\{n<\mathcal{U}\}} |\Phi(n)|\, dx \leq 2\mathit{\Pi}(t) + \pi \int_{\sqrt{\beta(a)}}^{\infty} |\Phi \circ \beta^{-1}(s)| ds + C_T, \tag{5.14}$$

for any $0 \leq t \leq T$*, and any* $T > 0$*.*

Previous lemma allows us to control the negative part of the internal energy once we know that the second moment is locally bounded in time.

Now, we still need to work on the differential equation [CPZ04, Per05] verified by $\mathit{\Pi}(t)$,

$$\frac{d}{dt}\mathit{\Pi}(t) = 2\int f(n)\, dx - \frac{\chi M^2}{4\pi}. \tag{5.15}$$

In order to estimate the first term, we propose to compare f and Φ near infinity to avoid the potential unboundedness of n in contrast to Lemma 5.10.

Hypothesis H5.13. *There exists* $\bar{\mathcal{U}}$ *such that* $f(u) \leq C\Phi(u)$ *for all* $u \geq \bar{\mathcal{U}}$*.*

We can now split the integral of $f(n)$ into three terms as

$$\frac{d}{dt}\mathit{\Pi}(t) \leq 2\int_{\{n<a\}} f(n)\, dx + 2\int_{\{a\leq n<\bar{\mathcal{U}}\}} f(n)\, dx + 2\int_{\{n\geq\bar{\mathcal{U}}\}} f(n)\, dx. \tag{5.16}$$

The first right-hand side term may be controlled as in the proof of Lemma 5.10 and thus, assuming hypothesis (H5.8), we deduce

$$\int_{\{n<a\}} f(n)\, dx \leq A_T\mathit{\Pi}(t) + B_T, \tag{5.17}$$

for any $0 \leq t \leq T$ and any $T > 0$.

We have already seen in Lemma 5.10 that the second term of (5.16) is easily bounded. In addition, thanks to assumption (H5.13), the free energy estimate (5.12) and a simple estimate of the integral on the set $\{a < n < \mathcal{U}\}$ as in Lemma 5.10, we conclude

$$\int_{\{n\geq\bar{\mathcal{U}}\}} f(n)dx \leq C\int_{\{n\geq\bar{\mathcal{U}}\}} \Phi(n)dx \leq \mathcal{E}_0 + C + C\int_{\{n<a\}} |\Phi(n)|dx,$$

and finally combining with (5.14), we get a very simple Gronwall type inequality

$$\frac{d}{dt}\mathit{\Pi}(t) \leq A_T + B_T\mathit{\Pi}(t), \tag{5.18}$$

for any $0 \leq t \leq T$ and any $T > 0$, which gives an *a priori* bound for the second moment of the cell density n.

Finally, coming back to the estimate (5.14) where we use that the second moment is locally in time bounded and going back to the free energy estimate (5.12), we finally conclude that

$$\int_{\mathbb{R}^2} \Phi^+(n)\,dx$$

is bounded for any $0 \leq t \leq T$ and any $T > 0$.

Remark 5.14. *The domination* (H5.13) *is valid as long as h has a power behaviour for large u, but fails if $h(u) = e^u$ for instance. However this dramatic situation is contained obviously in the assumptions of the previous section* 5.2.

Theorem 5.15 (Equi-integrability for the fast diffusion with an optimal constant)**.** *Assume* (H2.1), (H2.2), (H2.6), (H5.8), (H5.11) *and* (H5.13)*, then for all $T > 0$*

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\int_{\mathbb{R}^2}(n-k)_+\,dx = 0,$$

and therefore, $n \in L^\infty_{loc}\big(\mathbb{R}_+; L^\infty(\mathbb{R}^2)\big)$.

Note that in this proposition we only obtain local in time equi-integrability because of (5.14) and (5.18), on the contrary to theorems 5.3 and 5.5 where equi-integrability does not depend on time.

5.5 Conclusions of the *a priori* estimates

We remind the reader that we do not attempt here to make a complete existence theory for these models, but we remark that previous *a priori* estimates show that solutions obtained by suitable approximation procedures should satisfy uniform bounds on the cell density and then, the absence of blow-up in these models.

We can summarize the results of Sections 4 and 5, including Section 3 into the following main theorems:

Theorem 5.16 (No Blow-up: Bounded domain)**.** *Assume* (H2.1) *and* (H2.6) *with* Ω *bounded, then any solution* n *of* (2.1) *with initial data satisfying* $n_0 \in L^1_+(\Omega) \cap L^\infty(\Omega)$ *exists globally in time. Moreover, the cell density* n *is globally bounded in* L^∞*.*

Theorem 5.17 (No Blow-up: $\mathbb{R}^2$)**.** *Assume* (H2.1) *and* (H2.6) *and take any initial data satisfying* $n_0 \in L^1_+(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *such that the second moment of* n_0 *is finite. Then, we have the following three independent statements:*

(i) In addition, let us assume (H5.1)*, then any solution of* (2.3) *exists globally in time and the cell density* n *is uniformly bounded in time in* L^∞*.*

(ii) In addition, let us assume (H5.4) *and* (H5.8)*, then any solution of* (2.3) *exists globally in time and the cell density* n *is uniformly bounded in time in* L^∞*.*

(iii) In addition, let us assume (H5.8), (H5.11) *and* (H5.13)*, then any solution of* (2.3) *exists globally in time and the cell density* n *is locally bounded in time in* L^∞*.*

6 Extension to a nonlinear chemosensitivity

We plan to extend our previous results to both nonlinear diffusion and chemosensitivity $\chi(n)$. The first equation of our model is modified as following.

$$\partial_t n + \nabla \cdot \Big(- \nabla f(n) + \chi(n) n \nabla c \Big) = 0 \quad t \geq 0\,,\ x \in \Omega \subset \mathbb{R}^2. \tag{6.1}$$

First of all, we could keep all hypothesis made on h, and add some new hypothesis: basically χ is a positive bounded function. However, we point out that in [PH02], it comes from the derivation of the model that h and χ are linked by an underlying function q (1.4). Moreover, when we adapt previous arguments to this new system, it is natural to introduce a reduced diffusion term H, given by

$$H'(u) = \frac{f'(u)}{\chi(u)u} = \frac{h'(u)}{\chi(u)}, \tag{6.2}$$

which relates h and χ and plays in fact the role of h.

Although it seems that we have already captured the feature of this nonlinear system, there is a difficulty hidden in this additional nonlinearity. Because we assume (H2.6) that $uh'(u) \geq \delta$ for large u we reduce essentially to a linear diffusion in Section 3, and homogeneity is preserved in the calculations. But here χ may tend to zero: the diffusion is essentially nonlinear in the general case, and the Section 3 cannot be transposed exactly. That is why we will obtain by this method local in time estimates only (see section 6.2).

Let us now recall the hypothesis adapted to this new context.

Hypothesis HN 6.1 (Basic Regularity on the Nonlinear Reduced Diffusion H)**.** *$H \in L^1_{loc}[0,\infty) \cap \mathcal{C}^1(0,\infty)$ is an increasing function with $H(1) = 0$.*

We define without any ambiguity Φ and F corresponding to the functional H as in Section 2: $F'(u) = uH'(u)$ and $F(0) = 0$; $\Phi'(u) = H(u)$ and $\Phi(0) = 0$.

Hypothesis HN 6.2 (The Nonlinear Chemosensitivity)**.** *$\chi \in L^\infty(\mathbb{R}_+)$ is a positive function. We denote by χ_0 the bound $\|\chi\|_\infty$.*

Hypothesis HN 6.3 (Superlinear Reduced Diffusion at ∞)**.** *We define κ^*: $M = \mathcal{C}^*_{opt}\kappa^*$ as (1.5). We assume that H is growing faster than $\kappa^* \log u$ for large u, that is it exists $\kappa > \kappa^*$ and $\mathcal{U} \in \mathbb{R}_+$ such that*

$$\forall u \geq \mathcal{U} \quad H(u) \geq \kappa \log u.$$

Moreover we assume that it exists $\delta > 0$ such that $uH'(u) \geq \delta$ for large u.

Hypothesis HN6.4 (Degenerate Reduced Diffusion)**.** *We assume that $H(0^+) > -\infty$.*

Hypothesis HN 6.5 (Kowalczyk)**.** *It exists A such that $uH'(u) \geq \mathcal{H}$ for $u \geq A$; where $\mathcal{H} > \frac{3}{4}M\mathcal{C}_{gns}$.*

In the case of non-degenerate diffusion, Γ is defined as in section 5.3: $\Gamma(u) = \dfrac{F(u)}{u}$.

Hypothesis HN 6.6. *We assume that $F \circ \Gamma^{-1}$ is integrable near infinity.*

We organize this section into three subsections. We firstly look at the free energy, and what can be deduced by an energy method. Secondly, we adapt the Jäger & Luckhaus computations. Finally, we check the evolution of the second moment, to avoid loss of mass at infinity.

6.1 The free energy estimate

Thanks to the reduction (6.2) we get a free energy similar to Section 2.

Lemma 6.7 (Free energy). *Given a smooth solution of* (6.1)*, then the free energy functional* [CJM+01]

$$\mathcal{E}(t) = \int \Phi(n)\, dx - \frac{1}{2} \int nc\, dx \tag{6.3}$$

verifies

$$\frac{d}{dt}\mathcal{E} = -\int n\chi(n)\, |\nabla(H(n) - c)|^2\, dx \leq 0. \tag{6.4}$$

Proof. Indeed we can rewrite as following:

$$\partial_t n + \nabla \cdot \Big(\chi(n) n \big\{ - H'(n)\nabla n + \nabla c\big\}\Big) = 0.$$

We multiply by $H'\nabla n - \nabla c$, and we integrate by parts. Finally, we recover that the free energy

$$\mathcal{E}(t) = \int \Phi(n)\, dx - \frac{1}{2} \int nc\, dx,$$

is decreasing. □

At this stage, we could exactly reproduce the arguments in previous sections. However, we will see in the next subsections that this analogy is no longer valid for the whole analysis of (6.1).

Let us start by the simple cases which generalize to the present situation without any further difficulty. We can treat by the energy method the equi-integrability, both in the case of a bounded domain (Section 4), and a degenerate diffusion in the whole space (section 5.1).

Proposition 6.8 (Equi-integrability in Ω bounded). *Assume* (HN6.1), (H2.2), (HN6.2), (HN6.3) *then*

$$\lim_{K\to\infty} \sup_{t\geq 0} \int_\Omega (n - K)_+\, dx = 0,$$

Proposition 6.9 (Equi-integrability for degenerate diffusion). *Assume* (HN6.1), (H2.2), (HN6.2), (HN6.3) *and* (HN6.4)*, then*

$$\lim_{K\to\infty} \sup_{t\geq 0} \int_{\mathbb{R}^2} (n - K)_+\, dx = 0,$$

6.2 The Jäger & Luckhaus-type computation

We attempt here to reproduce and adapt the direct computation of $\frac{d}{dt}\int(n-K)_+^p\,dx$ as in sections 3 and 5.2. For this purpose, we fix a number K, and we deal with a convex functional $\varphi_K \in \mathcal{C}^2$ satisfying both $\varphi_K(K) = \varphi_K'(K) = 0$. Note that φ_K stands for the additional nonlinearity and takes the place of $(n-K)_+^p$. Then, we have

$$\begin{aligned}\frac{d}{dt}\int_{\{n\geq K\}}\varphi_K(n)\,dx &= \int_{\{n\geq K\}}\varphi_K'(n)\partial_t n\,dx \\ &= -\int_{\{n\geq K\}}\varphi_K''(n)\nabla(n-K)\cdot\Big(\chi(n)n\big\{H'(n)\nabla n-\nabla c\big\}\Big)\,dx,\end{aligned}$$

and thus,

$$\begin{aligned}\frac{d}{dt}\int_{\{n\geq K\}}\varphi_K(n)\,dx = &-\int_{\{n\geq K\}}\varphi_K''(n)\chi(n)nH'(n)|\nabla(n-K)|^2\,dx \\ &+\int_{\{n\geq K\}}\varphi_K''(n)\chi(n)n\nabla c\cdot\nabla(n-K)\,dx.\end{aligned}\tag{6.5}$$

In order to recover the background of the Section 3 (namely, the Gagliardo-Nirenberg-Sobolev inequality), we define precisely φ_K by

$$p(p-1)(v-K)_+^{p-2} = \varphi_K''(v)\chi(v),\tag{6.6}$$

in such a way that (6.5) becomes

$$\frac{d}{dt}\int_{\{n\geq K\}}\varphi_K(n)\,dx \leq -\eta\int(n-K)_+^p\,dx + O\Big(\int(n-K)_+\,dx\Big).\tag{6.7}$$

Remark 6.10. *Note that the calculations* (3.1), (3.2) *and* (3.3) *involve only the right-hand sides of the inequalities. This justifies the validity of* (6.7), *thanks to the choice* (6.6).

We are now able to explicit the difficulty hidden in the nonlinear chemosensitivity. It is not possible to deduce strictly from (6.7) that $\int(n-K)_+^p\,dx$ is bounded uniformly in time, because $\varphi_K(v)$ and $(v-K)_+^p$ have not the same homogeneity. However, we get that $\int\varphi_K(n)\,dx$ grows at most linearly in time. Moreover, we integrate twice (6.6) and we find that $(v-K)_+^p \leq \chi_0\varphi_K(v)\ \forall v\geq K$. As a consequence, we deduce

$$\int(n-K)_+^p\,dx \leq \chi_0\int_{\{n\geq K\}}\varphi_K(n)\,dx.$$

Lemma 6.11 (L^p bound from equi-integrability)**.** *Assume* (HN6.1), (HN6.2), (H2.2), (HN6.3). *In addition given $T > 0$, assume the modulus of equi-integrability ω is such that*

$$\omega(T, K) \xrightarrow[K \to \infty]{} 0.$$

Then $n \in L^\infty\big(0, T; L^p\big)$ for $p > 2$.

Remark 6.12. *Suppose in addition that the equi-integrability modulus does not depend on time. We integrate* (6.7) *in time and we apply the Gronwall lemma resulting into*

$$\frac{1}{t} \int_0^t \int (n - K)_+^p \, dx \, ds \in L^\infty(\mathbb{R}_+),$$

which is in a sense better than lemma 6.11*, but weaker than lemma* 3.1*.*

Remark 6.13. *If the positive function χ is bounded from below by a positive constant, and if the equi-integrability modulus does not depend on time, then the bound we are looking for is also uniform in time. In fact, this situation is essentially similar to the case of χ being constant.*

Next we examine the validity of the corresponding lemma 3.2.

Lemma 6.14 (L^∞ bound)**.** *Assume* (HN6.1), (HN6.2), (H2.2), (HN6.3), *and also that the chemotactic potential $\nabla c \in L^\infty_{loc}\big(\mathbb{R}_+; L^\infty\big)$, then the density satisfies $n \in L^\infty_{loc}\big(\mathbb{R}_+; L^\infty\big)$ too.*

Proof. We combine the proof of lemma 3.2 with (6.5) and (6.6) to obtain:

$$\begin{aligned}\frac{d}{dt} \int_{\{n \geq K\}} \varphi_K(n) \, dx \leq & - p^2 C \|\nabla c\|_\infty^2 \int (n - k)_+^p \, dx \\ & + C^2 p^4 \frac{\|\nabla c\|_\infty^4}{\delta^2} \left(\int (n - k)_+^{p/2} \, dx \right)^2 + p^2 C \|\nabla c\|_\infty^2,\end{aligned}$$

where the generic constant C does not depend on p. We integrate in time for $p = 2^j$, and we use (HN6.2) to get

$$\int (n - K)_+^{2^j} \, dx \leq C_T 2^{4j} K_{j-1}^2 + 2^{2j} C_T,$$

with $K_j = \sup_{t \in [0,T]} \int (n - k)_+^{2^j} \, dx$. This ensures that $n \in L^\infty\big(0, T; L^\infty\big)$. $\square$

On the other hand we check the validity of the theorem 5.5 in the case of a nonlinear chemosensitivity. By analogous arguments we obtain:

Proposition 6.15 (Equi-integrability for a non optimal constant)**.** *Assume* (HN6.1), (HN6.2), (H2.2), (HN6.3) *and* (HN6.5)*, then*

$$\forall T > 0 \quad \lim_{K\to\infty} \sup_{t\in[0,T]} \int_{\mathbb{R}^2} (n-K)_+ \, dx = 0.$$

6.3 Cell density control at ∞

When dealing with the model settled in the whole space, precise calculations of the second moment play a crucial role (see section 5.3).

Lemma 6.16 (Avoiding loss of mass at ∞)**.** *Assume* (HN6.1), (HN6.2) *and that the solution satisfies both* ∇c *and* $n \in L^\infty_{loc}\big(\mathbb{R}_+; L^\infty(\mathbb{R}^2)\big)$*. Moreover, assume either* (HN6.4) *or* (HN6.6)*, then* $\mathit{\Pi}(t) \in L^\infty_{loc}(\mathbb{R}_+)$.

Proof. We reproduce both the proofs of section 5.3. First we recover an inequality similar to (5.9),

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} \frac{|x|^2}{2} n(t,x)\,dx =& \int_{\mathbb{R}^2} f(n)\,dx + \int_{\mathbb{R}^2} \chi(n) n (x\cdot\nabla c)\,dx \\
\le& \chi_0 \int_{\mathbb{R}^2} F(n)\,dx + \chi_0 \int_{\mathbb{R}^2} n|x||\nabla c|\,dx \\
\le& \chi_0 \int_{\mathbb{R}^2} F(n)\,dx + \chi_0 \|\nabla c\|_\infty \Big(\int_{\mathbb{R}^2} n|x|^2\,dx\Big)^{1/2}\sqrt{M} \\
\le& \chi_0 \int_{\mathbb{R}^2} F(n)\,dx + \chi_0 \|\nabla c\|_\infty \sqrt{2M}\big(\mathit{\Pi}(t)\big)^{1/2}.
\end{aligned}$$

In the case of degenerate diffusion (HN6.4), we just control $\int F(n)$ as in lemma 5.7. In the case of fast diffusion together with (HN6.6) we get an estimate similar to (5.11), and we are able to conclude in the same way that $\mathit{\Pi}(t) \in L^\infty_{loc}(\mathbb{R}_+)$. □

Remark 6.17. *Theorem* 5.15 *cannot be generalized to this case. Due to the nonlinear chemoattractive feedback, we assumed a bound on* ∇c *which we are not able to combine with the calculations of section* 5.4.

6.4 Nonlinear diffusion and chemosensitivity results

We are now able to state the corresponding theorems to section 5.5, thanks to the combination of sections 6.1, 6.2 and 6.3. On a bounded domain, situation is quite similar, except the local in time estimate. Let us recall the equations we deal with:

$$\begin{cases} \partial_t n + \nabla \cdot \Big(- \nabla f(n) + \chi(n) n \nabla c \Big) = 0 \quad t \geq 0\,,\ x \in \Omega, \\ -\Delta c = n - < n >, \end{cases} \tag{6.8}$$

together with Neumann boundary conditions.

Theorem 6.18 (No finite-time Blow-up: Bounded domain)**.** *Assume* (HN6.1), (HN6.2) *and* (HN6.3) *with* Ω *bounded, then any solution* n *of* (6.8) *with initial data satisfying* $n_0 \in L^1_+(\Omega) \cap L^\infty(\Omega)$ *exists globally in time. Moreover, the cell density* n *lies in* $L^\infty_{loc}\big(\mathbb{R}_+; L^\infty(\Omega)\big)$*.*

Concerning the whole space we generalize theorem 5.17, except the last item.

$$\begin{cases} \partial_t n + \nabla \cdot \Big(- \nabla f(n) + \chi(n) n \nabla c \Big) = 0 \quad t \geq 0\,,\ x \in \mathbb{R}^2, \\ c(t,x) = -\dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \log|x-y|\, n(t,y)\, dy. \end{cases} \tag{6.9}$$

Theorem 6.19 (No finite-time Blow-up: $\mathbb{R}^2$)**.** *Assume* (HN6.1), (HN6.2) *and* (HN6.3)*. For any initial data satisfying* $n_0 \in L^1_+(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *such that the second moment of* n_0 *is finite, then the following independent statements hold:*

(i) In addition, we assume (HN6.4)*, then the solution of* (6.9) *exists globally in time and the cell density* n *is locally in time bounded in* $L^\infty(\mathbb{R}^2)$*.*

(ii) In addition, we assume (HN6.5) *and* (HN6.6)*, then the solution of* (6.9) *exists globally in time and the cell density* n *is locally in time bounded in* $L^\infty(\mathbb{R}^2)$*.*

Please note that these results are well adapted to examples mentioned above in 2.8.

1. The choice $q(u) = \frac{1}{1+u^\gamma}$ in (1.4) leads essentially to the linear reduced diffusion with coefficient $\frac{\kappa(\gamma+1)}{\chi_0}$. Because it corresponds to fast diffusion we have to distinguish between a bounded domain and the whole space: the threshold we found is respectively optimal (HN6.3) and non optimal (HN6.5).

2. The choice $q(u) = \exp(-\beta u)$ leads to a superlinear reduced diffusion, and solution is global in time either on a bounded domain or in the whole space.

These theorems also hold for the regularized system proposed by Velàzquez to understand what may happen after the blow-up time [Vel04a, Vel04b].

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