Blow-up, concentration phenomenon and global existence for the Keller-Segel model in high dimension

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Abstract

This paper is devoted to the analysis of the classical Keller-Segel system over $\mathbb{R}^d$, $d \geq 3$. We describe as much as possible the dynamics of the system characterized by various criteria, both in the parabolic-elliptic case and in the fully parabolic case. The main results in the parabolic-elliptic case are: local existence without smallness assumption on the initial density and a quantified blow-up rate, global existence under an improved smallness condition and comparison of blow-up criteria. A new concentration phenomenon for the fully parabolic case is also given.

Key words. Chemotaxis, parabolic systems, global weak solutions, local weak solutions, blow-up, energy methods.

AMS subject classification: 35B60; 35B44; 35Q92; 92C17; 92B05.

1 Introduction

This paper aims to describe the dynamics of the Keller-Segel system over $\mathbb{R}^d$, in high dimension $d \geq 3$, with a particular emphasis given to blow-up and related facts. The system reads as follows, in an adimensionalized formulation,

$$\left\{ \begin{array}{rcl} \partial_t n &=& \Delta n - \nabla \cdot (n \nabla c)\,, \\ \varepsilon\, \partial_t c &=& \Delta c + n - \alpha c\,. \end{array} \right. \tag{KS}$$

It describes, at the macroscopic scale, a population of cells with density n which attract themselves by secreting a diffusive chemical signal with concentration c. The nonnegative parameter ε is proportional to the ratio between the two diffusion coefficients of n and c, appearing in the dimensionalized formulation of (KS). It takes into account the different time scales of the two diffusion processes. The chemical degradation rate α is also a nonnegative constant. It is related to the range of action of the signal.

The choice $\varepsilon = 0$ describes the chemical concentration evolution in a quasi-stationary approximation. In this case, the chemical concentration is defined as:

$$c = \begin{cases} E_d * n\,, & \alpha = 0\,, \\ B_d^\alpha * n\,, & \alpha > 0\,, \end{cases} \tag{1.1}$$

where E_d and B_d^α are respectively the Green's function for the Poisson's equation in $\mathbb{R}^d$, and the Bessel kernel:

$$E_d(x) = \mu_d \, \frac{1}{|x|^{d-2}}\,, \qquad \mu_d = \frac{1}{(d-2)|\mathbb{S}^{d-1}|}\,, \tag{1.2}$$

$$B_d^\alpha(x) = \int_0^{+\infty} \frac{1}{(4\pi t)^{d/2}}\, e^{-\frac{|x|^2}{4t} - \alpha t}\, dt\,. \tag{1.3}$$

The system is endowed with non negative initial data n_0, and c_0 if $\varepsilon > 0$, and with fast decay conditions at infinity. Moreover, the initial cell density n_0 is supposed to be an integrable function, so that the total initial mass of cells M is conserved along time:

$$M = \int_{\mathbb{R}^d} n_0(x)\; dx = \int_{\mathbb{R}^d} n(x,t)\; dx\,,$$

System (KS) is naturally equipped with the following free energy:

$$\begin{aligned}\mathcal{E}[n,c](t) = \int_{\mathbb{R}^d} n(x,t)\log n(x,t)\, dx - \int_{\mathbb{R}^d} n(x,t)c(x,t)\, dx \\ + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla c(x,t)|^2\, dx + \frac{\alpha}{2}\int_{\mathbb{R}^d} c^2(x,t)\, dx\,,\end{aligned} \tag{1.4}$$

satisfying the dissipation equation

$$\frac{d}{dt}\mathcal{E}[n,c](t) = -\int_{\mathbb{R}^d} n(x,t)\left|\nabla\left(\log n(x,t) - c(x,t)\right)\right|^2\, dx - \varepsilon\int_{\mathbb{R}^d} \left|\partial_t c(x,t)\right|^2\, dx\,. \tag{1.5}$$

Under the quasi-stationary assumption $\varepsilon = 0$, the free energy (1.4) reduces to the difference between the entropy and the potential energy:

$$\mathcal{E}[n](t) = \int_{\mathbb{R}^d} n(x,t)\log n(x,t)\, dx - \frac{1}{2}\int_{\mathbb{R}^d} n(x,t)c(x,t)\, dx\,. \tag{1.6}$$

It is worth recalling that the free energy has been crucially used to obtain the existence of global solutions of the two-dimensional Keller-Segel system [6, 13, 16, 15, 24, 46]. Here, it shall be used fruitfully in the analysis of blow-up (Proposition 1.2) and concentration phenomenon (Theorem 1.3).

In this paper we prove several results describing the possible behaviours of solutions of system (KS). For the sake of clarity, in the sequel the parabolic-elliptic system ($\varepsilon = 0$) will be denoted PE, while the parabolic-parabolic system ($\varepsilon > 0$) will be denoted PP.

To begin, we improve the known existence results for the PE system. Local and global existence when $d \geq 3$, have been investigated either using the theory of weak solutions and the derivation of uniform L^p bounds based on Sobolev inequalities (see for instance [3, 22]), or using the theory of mild solutions based on dispersion estimates for the heat kernel (see for instance [5, 41]). Although the latter method is certainly more robust, we opt here for the former strategy. Indeed, paying very much attention to using sharp functional inequalities, this strategy enables us to quantify the threshold on $\|n_0\|_{L^{d/2}}$ ensuring global existence. Consequently, we are able to discuss the gap between global existence and obstruction to global existence. In addition, this strategy yields local existence without smallness condition on n_0 and allows to quantify the blow-up rate for the density n given in [41] (Proposition 3.1). Our existence results are summarized in the following Theorem. The constant in (1.7) is associated to a Gagliardo-Nirenberg inequality.

Theorem 1.1 (Global existence for the PE system)**.** *Let $d \geq 3$, $\alpha \geq 0$ and $\varepsilon = 0$. Let n_0 be a nonnegative initial density in $(L^1 \cap L^a)(\mathbb{R}^d)$, with $a > d/2$. Assume in addition that $n_0 \in L^1(\mathbb{R}^d, \psi(x)dx)$ where ψ is a nonnegative function such that $\psi(x) \to +\infty$ uniformly as $|x| \to +\infty$, $e^{-\psi} \in L^1(\mathbb{R}^d)$ and $\nabla\psi \in L^\infty(\mathbb{R}^d)$. Assume finally that*

$$\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} < \frac{8}{d}\, C_{GN}^{-2(1+\frac{2}{d})}\left(\frac{d}{2},\, d\right) . \tag{1.7}$$

Then, there exists a global weak nonnegative solution in the distribution sense (n, c) of (KS)*, where c is given by* (1.1)*.*

It appears that the estimates developed for the PE case can not be reproduced optimally in the PP case. Therefore, we are not able to obtain here a local existence result without any smallness conditions, nor to improve the known smallness condition for the global existence when $\varepsilon > 0$ (see [21, 40] for the global existence under smallness conditions and [36, 51] for the existence vs. blow-up issue on bounded domain).

The question of blow-up of solutions of (KS) has been a challenging issue in the field of mathematical biology. Indeed, in two dimensions, the blow-up can be interpreted as the capacity of a population of cells to self-organize (*e.g. Dictyostelium discoideum*), [19, 37, 39, 48]. This biological meaning of blow-up is lost in higher dimensions. Nevertheless, the occurrence of blow-up for the PE system in any dimensions $d \geq 2$ has been investigated by several authors, due to its intrinsic interest. Results are available either on a bounded domain [37, 4, 45, 30, 28, 29, 33, 51], or in the full space [22, 13, 44]. We refer also to the reviews [32, 35, 49]. Notice that blow-up cannot occur in dimension $d = 1$, unless the system is strongly modified [20, 17]. The case $d = 2$ has been thoroughly studied since it exhibits a remarkable dichotomy: either $M < 8\pi$ and the solution is global in time, or $M > 8\pi$ and the solution blows-up in finite time, if the second moment is initially finite. The borderline case $M = 8\pi$ is a compromise: solutions are global in time but concentrates as

$t \to \infty$, [8, 12]. Finally, particular solutions of the PE system which concentrate as Dirac masses have been constructed in the radially symmetric case using matching asymptotics [28].

In this paper, using the free energy (1.6), we prove finite time blow-up of solutions of the high dimensional PE system having small enough second moment. The obtained criterion is contained in [4] under some restrictions (star-shaped domain, negative energy) and it appears to better adapt to the case $\varepsilon > 0$ than the classical criterion (4.5), (see Theorem 1.3).

Proposition 1.2 (Blow-up for the PE system). *Let $\varepsilon = 0$, $\alpha \geq 0$, $d \geq 3$ and $a > \frac{d}{2}$. Assume that the second moment of the nonnegative initial density $n_0 \in (L^1 \cap L^a)(\mathbb{R}^d)$ is small in the following sense*

$$\int_{\mathbb{R}^d} |x|^2 n_0(x)\, dx < K_2(d)\, M^{1+\frac{2}{d}} \exp\left(-\frac{2}{dM}\,\mathcal{E}[n_0]\right), \tag{1.8}$$

where $K_2(d) := \frac{d}{2\pi}\, e^{-\frac{d}{d-2}}$. Then, the solution of the (KS) *system constructed in Proposition 3.1 blows up in finite time, that is the maximal time of existence T_{max} is finite and $\lim_{t \nearrow T_{max}} \|n(t)\|_{L^a(\mathbb{R}^d)} = +\infty$. Moreover, the blow-up condition* (1.8) *is not compatible with the smallness condition (3.9) for global existence.*

It is worth noticing that there still exists a gap between the available criteria (1.8) and (4.5) ensuring blow-up and the global existence Theorem 1.1. Moreover, the complementarity of criterion (1.8) and the known criterion (4.5) is far from being clear. In Section 4, we give some evidence that these two criteria are genuinely different. *However we believe that* (1.8) *is better than* (4.5) *(see discussion at the end of Section 4).* All these open issues show how the higher dimensional case is much singular than the two-dimensional case.

When $\varepsilon > 0$, the question of deriving a criterion ensuring finite time blow-up in the whole space is still open, as far as we know, (see [36] for the bounded domain case). In this paper, we give a weaker result going in that direction. Namely, we prove finite time concentration of the density n when ε is small. The main novelty is the use of the corrected energy given by

$$\mathcal{F}[n,c](t) = \log\left(\int_{\mathbb{R}^d} |x|^2 n(x,t)\, dx\right) + \frac{2}{dM}\mathcal{E}[n,c](t)\,. \tag{1.9}$$

Analyzing the time evolution of $\mathcal{F}[n,c](t)$, we can extend the blow-up criterion (1.8) to the PP system as follows.

Theorem 1.3 (Density concentration for the PP system). *Let $\varepsilon > 0$, $\alpha = 0$ and $d \geq 3$. Assume that the nonnegative initial densities (n_0, c_0) have finite energy $\mathcal{E}[n_0, c_0]$ and satisfy*

$$\int_{\mathbb{R}^d} |x|^2 n_0(x)\, dx < K_2(d)\;\; M^{1+\frac{2}{d}} \exp\left(-\frac{2}{dM}\mathcal{E}[n_0, c_0]\right) \exp\left(-\varepsilon^{\gamma}\right), \tag{1.10}$$

where the constant $K_2(d)$ is the same as in (1.8) *and $\gamma \in (0,1)$. Let (n,c) be a sufficently smooth solution of* (KS) *generated by (n_0, c_0) and T_{max} the maximal* *time of existence (possibly infinite). Then there exists a constant* $C(d) > 0$ *such that*

$$\sup_{t\in[0,T_{max})} \|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \geq \frac{2(d-2)}{C^2(d)}\,\varepsilon^{\gamma-1}\,. \tag{1.11}$$

We stress out a recent result of blow-up for the one dimensional PP system with suitable nonlinear diffusion of the density cells [20]. Nevertheless, the authors strongly use the property that the free energy is bounded from below, which does not hold true in our context, at least in the usual sense (see Lemma 5.2). This makes the analysis of the PP system with linear diffusion more difficult and justifies our weaker result in Theorem 1.3.

To conclude, it is worth to recall that radial solutions of the two dimensional PP system on a ball, developing Dirac singularity in finite time, have been constructed in [30, 31] by matching asymptotics. Moreover, blow-up results for the non-symmetric two dimensional PP system on a bounded domain also exist [33, 34, 50, 51]. However, these results crucially use the boundedness of the domain and it is not clear whether the solutions blow-up in finite time or concentrate in infinite time. The above Theorem 1.3 also cannot be adapted to the two dimensional case. To complicate the picture in this case, the authors in [7] showed the existence of positive forward self-similar solutions of (KS) with $\alpha = 0$, decaying to zero at infinity and having mass larger than 8π (which is no longer possible in the parabolic-elliptic case), see also [47] and the references therein.

The plan of the paper is the following. In Section 2 we list useful sharp functional inequalities. In Section 3 we address the existence issue for the PE system, giving quantitative thresholds. In Section 4 we derive the blow-up criterion (1.8) for the PE case, and we compare it with the known criterion (4.5). In Section 5 we prove the concentration result for the PP system.

2 Functional inequalities and preliminaries

The inequalities mentioned in the introduction and used throughout this paper are firstly the classical Sobolev inequality

$$\|f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq C_S(d)\|\nabla f\|_{L^2(\mathbb{R}^d)}\,, \qquad C_S^2(d) := \frac{4}{d(d-2)|\mathbb{S}^d|^{2/d}}\,, \tag{2.1}$$

and the following special case of the sharp Hardy-Littlewood-Sobolev inequality

$$\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x)|x-y|^{-\lambda}g(y)\;dxdy\right| \leq C_{HLS}(d,\lambda)\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)}\,, \tag{2.2}$$

for $p = \frac{2d}{2d-\lambda}$, $0 \leq \lambda < d$. The best constant $C_{HLS}(d,\lambda)$ has been obtained by Lieb [42, 43]. More specifically, because of (1.2) and (1.3), inequality (2.2) will be used in the sequel with $\lambda = d-2$. In this case (2.1) and (2.2) turn to be dual, as proved for instance in [43, Theorem 8.3], with

$$C_{HLS}(d,d-2) = \pi^{\frac{d}{2}-1}\Gamma^{-1}\left(\frac{d}{2}+1\right)\left(\frac{\Gamma(d)}{\Gamma(d/2)}\right)^{\frac{2}{d}} \ ,$$

so that the following relation can be established

$$C_S^2(d) = \mu_d\, C_{HLS}(d,d-2)\ . \tag{2.3}$$

Let us observe that when $\lambda \searrow 0$, (2.2) boils down to the logarithmic Hardy-Littlewood-Sobolev inequality [18, 2] used in the two dimensional case [13, 16].

Next come two interpolation lemmas, useful in the sequel for the control of both the entropy $\int_{\mathbb{R}^d} n\log n$ and the potential $\int_{\mathbb{R}^d} nc$ in the free energy.

Lemma 2.1 (Entropy lower bound)**.** *Let f be any nonnegative $L^1(\mathbb{R}^d)$ function such that $I = \int_{\mathbb{R}^d}|x|^2 f(x)dx < \infty$ and $\int_{\mathbb{R}^d} f\log f < \infty$. Let $M = \int_{\mathbb{R}^d} f$. Then,*

$$\int_{\mathbb{R}^d} f\log f + \frac{dM}{2}(\log I + 1) \geq M\log M + \frac{dM}{2}\log\left(\frac{dM}{2\pi}\right)\ . \tag{2.4}$$

Proof. Let $\delta > 0$. Applying the Jensen's inequality with the probability density $\mu(x) = \delta^{d/2}\pi^{-d/2}e^{-\delta|x|^2}$, one obtain:

$$\int_{\mathbb{R}^d} f\log f = \int_{\mathbb{R}^d}\frac{f}{\mu}\log\left(\frac{f}{\mu}\right)\mu + \int_{\mathbb{R}^d} f\log\mu \geq M\log M + \int_{\mathbb{R}^d} f\log\mu\,,$$

i.e.

$$\int_{\mathbb{R}^d} f\log f + \delta I \geq M\log M + \frac{dM}{2}\log(\delta\,\pi^{-1})\,. \tag{2.5}$$

The optimization of (2.5) with respect to $\delta > 0$ under the fixed constrains I and M, yields (2.4), the optimal δ being $\delta = \frac{dM}{2I}$. □

Lemma 2.2 (Potential confinement)**.** *Let f be any nonnegative function such that $f \in (L^1 \cap L^{\frac{d}{2}})(\mathbb{R}^d)$ and $I = \int_{\mathbb{R}^d}|x|^2 f(x)\,dx < \infty$. Let $M = \int_{\mathbb{R}^d} f(x)\,dx$. Then,*

$$2^{1-\frac{d}{2}}M^{\frac{d}{2}+1}I^{1-\frac{d}{2}} \leq \iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f(x)\,f(y)}{|x-y|^{d-2}}\,dxdy \leq C_{HLS}(d,d-2)M\|f\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}. \tag{2.6}$$

Proof. The right inequality is a direct consequence of the Hardy-Littlewood-Sobolev inequality (2.2) and of standard interpolation (see also [11]). For the left inequality see [4]. □

We conclude this section by recalling some few basic properties satisfied by the Bessel kernel B_d^α defined in (1.3) and useful in the sequel. Since the authors do not found any references for them, the required properties are listed and proved in the lemma below.

Lemma 2.3 (Properties of the Bessel kernel)**.** *The following relations for B_d^α and ∇B_d^α hold true for $\alpha \geq 0$ in any space dimension $d \geq 3$.*

(i) Expansion of B_d^α with respect to E_d: $\quad B_d^\alpha = E_d - \alpha\, E_d * B_d^\alpha \quad$ *a.e..*

(ii) Gradient formula: $\quad \nabla B_d^\alpha(x) = -\frac{1}{|\mathbb{S}^{d-1}|}\,\frac{x}{|x|^d}\, g_\alpha(|x|)\,,$

where $g_\alpha(|x|) := \Gamma(d/2)^{-1}\int_0^{+\infty} s^{\frac{d}{2}-1} e^{-s-\alpha\frac{|x|^2}{4s}}\, ds$*.*

(iii) Corrected Euler's homogeneous function theorem:

$$x\cdot\nabla B_d^\alpha(x) = -(d-2)B_d^\alpha(x) - 2\alpha\,(B_d^\alpha * B_d^\alpha)(x)\,. \tag{2.7}$$

Proof. The convolutions in (i) and in (2.7) are well defined since both B_d^α and E_d belong to $L_w^{\frac{d}{d-2}}(\mathbb{R}^d)$, and $B_d^\alpha \in L^p(\mathbb{R}^d)$ for $1 \leq p < \frac{d}{d-2}$ [43]. The expansion of B_d^α in (i) is a straightforward computation in Fourier variable. Alternatively, notice that $E_d * B_d^\alpha$ is the unique solution of $(-\Delta+\alpha)(E_d * B_d^\alpha) = E_d$ belonging to $L^r(\mathbb{R}^d)$ for some $r \geq 1$, ([43]). The gradient formula in (ii) is a straightforward computation. To prove (2.7), first notice that the Fourier transform of B_d^α is $\left(\alpha + 4\pi^2|\xi|^2\right)^{-1}$, when defining the Fourier transform as $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\, x\cdot\xi} f(x)\, dx$, ([43]). Therefore, we have

$$(x\cdot\nabla_x B_d^\alpha)\hat{}(\xi) = -\nabla_\xi\cdot\left(\frac{\xi}{\alpha+4\pi^2|\xi|^2}\right) = -\frac{d-2}{\alpha+4\pi^2|\xi|^2} - \frac{2\alpha}{\left(\alpha+4\pi^2|\xi|^2\right)^2}$$

and identity (2.7) is proved. This identity is also a direct consequence of the scaling property of the Bessel kernel $B_d^\alpha(\lambda x) = \lambda^{2-d} B_d^{\alpha\lambda^2}(x)$. $\square$

3 Local and global existence for the PE system

This section is devoted to the proof of the existence results for the PE system. We show essentially how to control the L^p norm of the density n, either locally in time, or globally when $\|n_0\|_{L^{d/2}}$ is small. We pay much attention to the quantitative value of the threshold ensuring global existence.

We state first a result of local existence that gives as a side effect a characterization of blow-up in terms of L^p norms of the density n.

Proposition 3.1. *Under the same hypotheses as Theorem 1.1 except the smallness condition* (1.7)*, there exists $T_{max} > 0$ depending only on $\|n_0\|_{L^a(\mathbb{R}^d)}$ and* *a weak nonnegative solution in the distribution sense (n, c) of* (KS)*, where c is given by* (1.1)*, such that*

$$n \in L^\infty((0, T_{max}); (L^1 \cap L^a)(\mathbb{R}^d)) \quad \textit{and} \quad n \in L^\infty_{\text{loc}}((0, T_{max}); L^p(\mathbb{R}^d)),$$

for all $p \in (a, \infty)$. Moreover, $n \in L^\infty_{\text{loc}}((0, T_{max}); L^1(\mathbb{R}^d, \psi(x)\,dx))$. The energy $\mathcal{E}[n] \in L^\infty_{\text{loc}}(0, T_{max})$, while the dissipation of energy $\int_{\mathbb{R}^d} n|\nabla(\log n - c)|^2 \in L^1_{\text{loc}}(0, T_{max})$. Finally, for $C(a, d)$ defined in (3.8)*, it holds true*

$$T_{max} \geq C(a, d)\, \|n_0\|_{L^a(\mathbb{R}^d)}^{\frac{2a}{d-2a}}, \tag{3.1}$$

and if $T_{max} < \infty$ then $\lim_{t \nearrow T_{max}} \|n(t)\|_{L^a(\mathbb{R}^d)} = +\infty$, with blow-up rate at least $\left(\frac{C(a,d)}{T_{max}-t}\right)^{\frac{2a-d}{2a}}$.

Proof. We give only the main a priori estimates below. The full regularization procedure is classical, and the details are provided in the Appendix.

Using the Gagliardo-Nirenberg inequality

$$\|v\|_{L^{\frac{2(p+1)}{p}}(\mathbb{R}^d)} \leq C_{GN}(p, d) \|\nabla v\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2(p+1)}} \ \|v\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{2(p+1)}}, \tag{3.2}$$

for $v = n^{\frac{p}{2}}$, we get for $p > \frac{d}{2}$ and $\delta > 0$ to be chosen later

$$\int_{\mathbb{R}^d} n^{p+1} \leq \frac{\delta^{r'}}{r'} \|n\|_{L^p(\mathbb{R}^d)}^{(p+1-\frac{d}{2})r'} + \frac{1}{r\delta^r}\, \lambda(p, d) \|\nabla n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^d)}^2, \tag{3.3}$$

where $r = \frac{2p}{d}$, $r' = \frac{r}{r-1}$ and $\lambda = C_{GN}^{\frac{4(p+1)}{d}}$. Plugging estimate (3.3) into the evolution equation for the L^p norm of n, one obtains

$$\frac{d}{dt} \int_{\mathbb{R}^d} n^p \leq \frac{(p-1)}{p} \left[\frac{d\lambda}{2\delta^r} - 4\right] \|\nabla n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^d)}^2 + (p-1) \frac{\delta^{r'}}{r'} \left(\int_{\mathbb{R}^d} n^p\right)^{1+(p-\frac{d}{2})^{-1}}. \tag{3.4}$$

Then, it is enough to choose $\delta = \delta(p) > 0$ such that $\left(\frac{d\lambda}{2\delta^r} - 4\right) \leq 0$, to have

$$\int_{\mathbb{R}^d} n^p(t) \leq h_p(t) \int_{\mathbb{R}^d} n_0^p, \qquad t \in [0, T_p), \tag{3.5}$$

if $n_0 \in L^p(\mathbb{R}^d)$, where

$$h_p(t) := \left[1 - \left(1 - \frac{1}{p}\right) \delta^{r'}\, t \left(\int_{\mathbb{R}^d} n_0^p\right)^{(p-\frac{d}{2})^{-1}}\right]^{(\frac{d}{2}-p)}, \tag{3.6}$$

and $T_p := p(p-1)^{-1} \delta^{-r'} \left(\int_{\mathbb{R}^d} n_0^p\right)^{(\frac{d}{2}-p)^{-1}}$.

The above argument can be bootstrapped in the following way. Let us choose δ the smallest possible, i.e. $\delta := (d\lambda 8^{-1})^{d/2p}$. Then, (3.5)-(3.6) reads as

$$\int_{\mathbb{R}^d} n^p(t) \leq \left(\frac{C(p,d)}{T_p - t}\right)^{p-\frac{d}{2}}, \qquad T_p = C(p,d)\|n_0\|_{L^p(\mathbb{R}^d)}^{\frac{2p}{d-2p}}, \tag{3.7}$$

with

$$C(p,d) := \frac{p}{p-1}\left(\frac{8}{d\lambda}\right)^{\frac{d}{2p-d}}. \tag{3.8}$$

Let $t_0 = 0$, $T_p^0(t_0) = T_p$ and let $t_1 < T_p^0(t_0)$ be such that (if it exists) a strict inequality in (3.7) holds true for t_1. Integrating (3.4) over (t_1, t) with the above δ, we obtain a new maximal time of existence $T_p^1(t_1) = t_1 + C(p,d)\,\|n(t_1)\|_{L^p(\mathbb{R}^d)}^{\frac{2p}{d-2p}}$, greater than T_p^0 and such that

$$\int_{\mathbb{R}^d} n^p(t) \leq \left(\frac{C(p,d)}{T_p^1(t_1) - t}\right)^{p-\frac{d}{2}}, \qquad t \in [t_1, T_p(t_1))\,.$$

Bootstrapping again, we can construct a strictly increasing sequence $T_p^k(t_k)$, whose upper limit is the maximal time of existence T_{max} satisfying (3.1). If $T_{max} < \infty$, the blow-up rate of n is an immediate consequence of the previous estimates. □

Proof of Theorem 1.1. In order to obtain the global existence result, we slightly modify the previous argument. Instead of the interpolation inequality (3.3), we plug the sharp Gagliardo-Nirenberg inequality (3.2) with $p = \frac{d}{2}$ into the evolution equation for the $L^{\frac{d}{2}}$ norm of n. Therefore, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} n^{\frac{d}{2}}(t) \leq \left(\frac{d}{2} - 1\right)\left[C_{GN}^{2(1+\frac{2}{d})}\left(\frac{d}{2}, d\right)\|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{8}{d}\right]\|\nabla n^{\frac{d}{4}}\|_{L^2(\mathbb{R}^d)}^2\,.$$

Under the hypothesis (1.7), the $L^{\frac{d}{2}}$ norm of n decreases for all times $t \geq 0$. □

Remark 3.2 (Improved smallness condition) In [22], in order to obtain the global existence result, the authors have used the Sobolev inequality (2.1) instead of (3.2), so they ended up with the corresponding smallness condition

$$\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} < \frac{8}{d\,C_S^2(d)}\,. \tag{3.9}$$

Since we can prove that $C_{GN}^{2(1+\frac{2}{d})}\left(\frac{d}{2}, d\right) < C_S^2(d)$, condition (1.7) is a weaker condition than (3.9). Moreover, inequality (3.2) is sharp (equality holds true for the extremal functions).

Remark 3.3 It is easy to prove that under the smallness condition (3.9), the lower bound for T_{max} in (3.1) tends to ∞, showing that the estimates are sharp. Indeed, (3.3) can be obtained also using the Sobolev inequality (2.1) as follows

$$\int_{\mathbb{R}^d} n^{p+1} \leq \|n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^d)}^{(p+1-\frac{d}{2})\frac{2}{p}}\,\|n^{\frac{p}{2}}\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^{\frac{d}{p}} \leq \|n\|_{L^p(\mathbb{R}^d)}^{(p+1-\frac{d}{2})}\left(C_S(d)\|\nabla n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^d)}\right)^{\frac{d}{p}}.$$

Therefore, $\lambda(p,d) \leq C_S^2(d)$. Defining $\eta(d) := d\, C_S^2(d) 8^{-1} \|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$, the lower bound (3.1) becomes: $T_{max} \geq \frac{a}{a-1} \eta^{\frac{d}{d-2a}} \left(\frac{\int n_0^{\frac{d}{2}}}{\int n_0^a} \right)^{\frac{1}{a-d/2}}$ and it tends to ∞ as $a \to \frac{d}{2}$, whenever $\eta(d) < 1$, since $\left(\frac{\int n_0^{\frac{d}{2}}}{\int n_0^a} \right)^{\frac{1}{a-d/2}} \leq M^{\frac{1}{a-1}} (\int n_0^a)^{-\frac{1}{a-1}}$.
When considering the improved smallness condition (1.7), (3.1) reads as

$$T_{max} \geq \frac{a}{a-1} \eta^{\frac{d}{d-2a}} \left(\frac{\int n_0^{\frac{d}{2}}}{\int n_0^a} \right)^{\frac{1}{a-d/2}} \left[C_{GN}^{2(1+\frac{2}{d})} \left(\frac{d}{2}, d \right) \lambda^{-1}(a,d) \right]^{\frac{d}{2a-d}} , \tag{3.10}$$

where $\eta = d 8^{-1} C_{GN}^{2(1+\frac{2}{d})} \left(\frac{d}{2}, d \right) \|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$. Then, the previous limit holds true if the limit of the bracket in (3.10) stays positive as $a \to \frac{d}{2}$. In that direction, we observe that $\lambda(a,d) = C_{GN}^{\frac{4(a+1)}{d}}(a,d) \to C_{GN}^{2(1+\frac{2}{d})} \left(\frac{d}{2}, d \right)$ as $a \to \frac{d}{2}$, being the constant C_{GN} continuous in p, [23].

4 Blow-up for the PE system

In order to derive a blow-up criterion for solutions of the PE system, the general idea is to follow the evolution of the second moment $I(t) := \int_{\mathbb{R}^d} |x|^2 n(x,t)\, dx$. Namely, one aims to prove that I satisfies a differential inequality like

$$\frac{d}{dt} I(t) \leq f(I(t)) , \tag{4.1}$$

where f is a continuous nondecreasing function such that $f(0) < 0$. This clearly ensures blow-up when $I(0) < I^*$, where $I^* := \inf\{I > 0 \,|\, f(I) = 0\}$.

The above technique as been applied firstly by Biler in [4] for a model of gravitational interaction of particles on a star-shaped domain of $\mathbb{R}^d$, similar to the (KS) system with $\varepsilon = \alpha = 0$. Successively, it has been used by several authors in the context of the Keller-Segel system (see for instance [45, 22, 13]). The methodology is also reminiscent of the blow-up criteria for the nonlinear Schrödinger equation initiated by Glassey [25], and has been successively applied to kinetic gravitational models [26] and kinetic chemotaxis models [14].

That approach gives sharp results in two dimensions. In the case $d \geq 3$, the derivation of (4.1) is more complicated since the potential appears in the evolution equation for $I(t)$. More precisely, from

$$\frac{d}{dt} I(t) = 2dM + 2 \int_{\mathbb{R}^d} n(x,t)\, x \cdot \nabla c(x,t)\, dx , \tag{4.2}$$

we have (after symmetrization of the integral term) for $\alpha = 0$

$$\frac{d}{dt} I(t) = 2dM - \frac{1}{|\mathbb{S}^{d-1}|} \iint_{\mathbb{R}^d \times \mathbb{R}^d} n(x,t) \frac{1}{|x-y|^{d-2}} n(y,t)\, dxdy , \tag{4.3}$$

while for $\alpha > 0$

$$\frac{d}{dt}I(t) = 2dM - \frac{1}{|\mathbb{S}^{d-1}|}\iint_{\mathbb{R}^d\times\mathbb{R}^d} n(x,t)\frac{g_\alpha(|x-y|)}{|x-y|^{d-2}}n(y,t)\,dxdy\,, \tag{4.4}$$

where g_α is defined in Lemma 2.3. Therefore, different blow-up criteria can be obtained according to how the right hand side of (4.3) or (4.4) is estimated with respect to I, $\mathcal{E}[n]$ and M.

For instance, one can obtain the following blow-up criterion

$$\int_{\mathbb{R}^d} |x|^2 n_0(x)\,dx < K_1^\alpha(d,M)M^{\frac{d}{d-2}}\,, \tag{4.5}$$

where $K_1^\alpha(d,M)$ is defined in (4.6), $K_1^0(d,M) = K_1(d) := 2^{-\frac{d}{d-2}}(d|\mathbb{S}^{d-1}|)^{-\frac{2}{d-2}}$, (see [4, 45, 9, 22] for $\alpha = 0$), using (2.6) into (4.3), to get

$$\frac{d}{dt}I(t) \le 2dM - |\mathbb{S}^{d-1}|^{-1}2^{\,1-\frac{d}{2}}\,M^{\,\frac{d}{2}+1}\,I^{\,1-\frac{d}{2}}(t)\,.$$

Accordingly we get an obstruction to global existence when $I(0) < K_1(d)\,M^{\frac{d}{d-2}}$. We can improve this criterion in the case $\alpha > 0$. Using the non-decreasing behaviour of g_α, we have for any $R > 0$,

$$\begin{aligned}\frac{d}{dt}I(t) &\le 2dM - \frac{g_\alpha(R)}{|\mathbb{S}^{d-1}|}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{n(x,t)n(y,t)}{|x-y|^{d-2}}dxdy + \frac{g_\alpha(R)}{|\mathbb{S}^{d-1}|}\iint\limits_{|x-y|>R}\frac{n(x,t)n(y,t)}{|x-y|^{d-2}}dxdy\,,\\ &\le 2dM - \frac{g_\alpha(R)}{|\mathbb{S}^{d-1}|}\,2^{1-\frac{d}{2}}M^{\,\frac{d}{2}+1}I^{\,1-\frac{d}{2}}(t) + \frac{g_\alpha(R)}{|\mathbb{S}^{d-1}|\,R^{d-2}}M^2\,.\end{aligned}$$

Optimizing with respect to R, we necessarily have blow-up when $I(0)$ is less than $K_1^\alpha(d,M)M^{\frac{d}{d-2}}$, where

$$K_1^\alpha(d,M) = \frac{1}{2}\left[\sup_{R>0}\left(\frac{g_\alpha(R)\,R^{d-2}}{2d|\mathbb{S}^{d-1}|\,R^{d-2}+g_\alpha(R)\,M}\right)\right]^{\frac{2}{d-2}}\,. \tag{4.6}$$

Since $g_\alpha(R)$ is positive and exponentially decreasing to 0 as $R\to\infty$, K_1^α is well defined. We check that $K_1^0(d,M) = K_1(d)$ because $g_0(|x|)\equiv 1$ and the supremum in (4.6) is achieved for $R\to+\infty$.

It is instructive to check incompatibility between (4.5) and the smallness condition (3.9). It is sufficient to consider only the case $\alpha = 0$, since K_1^α is strictly decreasing with respect to α. From the potential confinement Lemma 2.2 and from (4.5), we have

$$\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \ge 2^{1-\frac{d}{2}}\,M^{\frac{d}{2}}\,C_{HLS}^{-1}(d,d-2)\,I^{1-\frac{d}{2}}(0) > 2\,d\,|\mathbb{S}^{d-1}|\,C_{HLS}^{-1}(d,d-2)\;. \tag{4.7}$$

Using the relation (2.3) into (4.7), the reverse of condition (3.9) follows.

Concerning the blow-up criterion (1.8), the proof is given in the next subsection.

4.1 Proof of Proposition 1.2

We first consider the case $\alpha = 0$. Since the free energy is non-increasing, we get using (2.4),

$$\begin{aligned}\frac{d}{dt}I(t) &= 2dM + 2(d-2)\left(\mathcal{E}[n](t) - \int_{\mathbb{R}^d} n(x,t)\log n(x,t)\,dx\right) \\ &\le d(d-2)M\log I(t) + 2(d-2)\mathcal{E}[n_0] + B(d,M)\,,\end{aligned} \tag{4.8}$$

where the constant $B(d,M)$ is defined as

$$B(d,M) = d^2M - 2(d-2)\left[M\log M + \frac{dM}{2}\log\left(\frac{dM}{2\pi}\right)\right]. \tag{4.9}$$

Finite time blow-up follows when $I(0) < \exp\left(-\frac{2}{dM}\mathcal{E}[n_0] - \frac{B(d,M)}{d(d-2)M}\right)$, i.e. (1.8).

In the case $\alpha > 0$, we combine symmetrization of the integral term in (4.2) and formula (2.7), to deduce

$$\begin{aligned}\frac{d}{dt}I(t) &= 2dM - (d-2)\iint_{\mathbb{R}^d\times\mathbb{R}^d} n(x,t)\,B_d^\alpha(x-y)\,n(y,t)\,dxdy \\ &\quad -2\alpha\iint_{\mathbb{R}^d\times\mathbb{R}^d} n(x,t)(B_d^\alpha * B_d^\alpha)(x-y)\,n(y,t)\,dxdy \\ &= 2dM - (d-2)\int_{\mathbb{R}^d} n(x,t)c(x,t)dx - 2\alpha\int_{\mathbb{R}^d}(B_d^\alpha * n)^2(x,t)dx \\ &= 2dM + 2(d-2)\left(\mathcal{E}[n](t) - \int_{\mathbb{R}^d} n(x,t)\log n(x,t)\,dx\right) - 2\alpha\int_{\mathbb{R}^d} c^2(x,t)\,dx.\end{aligned}$$

Neglecting the last negative contribution, we are reduced to the previous estimate (4.8) and we can conclude as in the case $\alpha = 0$.

In order to check the incompatibility of this criterion with the global existence threshold (3.9), let us rewrite (1.8) as

$$\begin{aligned}\frac{dM}{2}(\log I(0) + 1) + \int_{\mathbb{R}^d} n_0(x)\log n_0(x)\,dx - M\log M - \frac{dM}{2}\log\left(\frac{dM}{2\pi}\right) \\ < \frac{1}{2}\int_{\mathbb{R}^d} n_0(x)c_0(x)\,dx - \frac{dM}{d-2}\,.\end{aligned} \tag{4.10}$$

The left hand side of (4.10) is nonnegative owing to Lemma 2.1. Then, from (4.10) and the potential confinement Lemma 2.2, we have

$$\frac{2dM}{d-2} < \int_{\mathbb{R}^d} n_0(x)c_0(x)\,dx \le \mu_d\, C_{HLS}(d,d-2)\,M\,\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}\,,$$

and the reverse of condition (3.9) follows.

Remark 4.1 Coming back to the line (4.8) and using the fact that $\frac{d}{dt}\mathcal{E}[n(t)] \le 0$ we deduce the following differential inequality for the corrected energy defined in (1.9)

$$I(t)\frac{d}{dt}\mathcal{F}[n](t) \le d(d-2)M\mathcal{F}[n](t) + B(d,M)\,, \tag{4.11}$$

Moreover, the blow-up criterion involving the initial free energy (1.8) reads equivalently as:

$$d(d-2)M\mathcal{F}[n_0] + B(d,M) < 0\,. \tag{4.12}$$

In Section 5, we will generalize the differential inequality (4.11) as well as the blow-up condition (4.12) to the PP system, in order to obtain a concentration result for the $L^{\frac{d}{2}}$ norm of n, (see (5.4) and (5.6) respectively).

Remark 4.2 It is easy to check the invariance of criteria (4.5) and (1.8) under the scaling $n_0(x) \to n_0^\lambda(x) = \lambda^{-2} n_0(\lambda^{-1}x)$, preserving the $L^{\frac{d}{2}}$–norm. Concerning criterion (4.5), one has to observe that when rescaling, α needs to be changed into α/λ^2 and that $K_1^{\alpha/\lambda^2}(d, \lambda^{d-2}M) = K_1^\alpha(d,M)$, thanks to the scaling property of g_α, i.e. $g_\alpha(|\frac{x}{\lambda}|) = g_{\alpha/\lambda^2}(|x|)$.

4.2 Complementarity of the two criteria (1.8) and (4.5)

We shall prove in this Section that the two blow-up criteria are in fact complementary when $\alpha = 0$, i.e. *none of the criteria* (4.5) *and* (1.8) *contain the other*. The case $\alpha > 0$ is not considered for sake of simplicity.

We first construct some initial datum n_0 satisfying

$$K_1(d)\, M^{\frac{d}{d-2}} < \int_{\mathbb{R}^d} |x|^2 n_0(x)\, dx < K_2(d)\, M^{1+\frac{2}{d}} \exp\left(-\frac{2}{dM}\,\mathcal{E}[n_0]\right)\,. \tag{4.13}$$

This is a direct consequence of the fact that the energy is unbounded from below. Let us introduce the following family of densities indexed by $\lambda > 0$,

$$n_0^\lambda(x) := \frac{1}{2}\left[\lambda^{-d}\varphi\left(\frac{x-a}{\lambda}\right) + \lambda^{-d}\varphi\left(\frac{x+a}{\lambda}\right)\right]\,,$$

where $a \neq 0$ is some point to be chosen later and φ is a nonnegative function in $(L^1 \cap L^{\frac{d}{2}})(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \varphi(z)\, dz = M$ and $\operatorname{Supp}\varphi \subset B(0,1)$. The densities n_0^λ belong to $(L^1 \cap L^{\frac{d}{2}})(\mathbb{R}^d)$, have mass equal to M, $L^{\frac{d}{2}}$-norm increasing as $\lambda \searrow 0$ and the second moment given by

$$\int_{\mathbb{R}^d} |x|^2 n_0^\lambda(x)\, dx = M|a|^2 + \lambda^2 \int_{\mathbb{R}^d} |z|^2 \varphi(z)\, dz\,. \tag{4.14}$$

When evaluating the free energy (1.6), the cross-interaction between the two densities located around a and $-a$ is zero in the entropy term, if λ is small enough so that the supports $B(a,\lambda)$ and $B(-a,\lambda)$ are disjoints. Hence, we have

$$\begin{aligned}\mathcal{E}[n_0^\lambda] = \int_{\mathbb{R}^d} \varphi(z) \log\varphi(z)\, dz - dM\log\lambda - M\log 2 - \frac{\mu_d}{4\lambda^{d-2}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\varphi(z)\,\varphi(z')}{|z-z'|^{d-2}} dz dz' \\ - \frac{\mu_d}{4} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\varphi(z)\,\varphi(z')}{|2a + \lambda(z-z')|^{d-2}}\, dz dz'\,,\end{aligned} \tag{4.15}$$

which goes to $-\infty$ as $\lambda \searrow 0$. Comparing (4.14) and (4.15) clearly there exists $a \neq 0$ and $\lambda > 0$ sufficiently small such that the corresponding density n_0^λ satisfies (4.13).

Next we construct an initial datum satisfying

$$K_2(d)\, M^{1+\frac{2}{d}} \exp\left(-\frac{2}{dM}\,\mathcal{E}[n_0]\right) < \int_{\mathbb{R}^d} |x|^2 n_0(x)\, dx < K_1(d)\, M^{\frac{d}{d-2}}\,. \qquad (4.16)$$

We shall guarantee that the entropy term dominates the potential term in the free energy $\mathcal{E}[n_0]$. For that purpose we consider the following sequence of densities

$$n_0^\lambda(x) := \frac{1}{N}\sum_{i=1}^{N} \lambda^{-d}\varphi\left(\frac{x-a_i}{\lambda}\right)\,, \qquad \lambda = N^{1/(2-d)}\,, \qquad (4.17)$$

where φ is defined as above and the family of points $(a_i)_{1\le i\le N}$ is symmetric (a_i and $-a_i$ belong both to the family). Again, the densities n_0^λ belong to $(L^1\cap L^{\frac{d}{2}})(\mathbb{R}^d)$, have mass equal to M, $L^{\frac{d}{2}}$-norm increasing as $\lambda \searrow 0$ and the second moment given by

$$\int_{\mathbb{R}^d} |x|^2 n_0^\lambda(x)\, dx = M\,\frac{1}{N}\sum_{i=1}^{N} |a_i|^2 + \lambda^2 \int_{\mathbb{R}^d} |z|^2 \varphi(z)\, dz\,.$$

Again we assume that λ is chosen such that the supports $B(a_i,\lambda)$ of each contribution in (4.17) are disjoints and we introduce the notation $D^\lambda(i,j) = \mathrm{dist}(B(a_i,\lambda), B(a_j,\lambda))$. Then, computing separately each contribution of the energy functional, we obtain

$$\int_{\mathbb{R}^d} n_0^\lambda(x)\log n_0^\lambda(x)\, dx = -M\log(N\lambda^d) + \int_{\mathbb{R}^d} \varphi(z)\log\varphi(z)\; dz\,, \qquad (4.18)$$

and

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} n_0^\lambda(x)\,\frac{1}{|x-y|^{d-2}}\,n_0^\lambda(y)\,dxdy &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\varphi(z)\,\varphi(z')}{|z-z'|^{d-2}}\,dzdz' \\ &\quad + \frac{1}{N^2}\sum_{i\neq j}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\varphi(z)\,\varphi(z')}{|a_i-a_j+\lambda(z-z')|^{d-2}}\,dzdz' \\ &\le \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\varphi(z)\,\varphi(z')}{|z-z'|^{d-2}}\,dzdz' + \frac{M^2}{N^2}\sum_{i\neq j}(D^\lambda(i,j))^{2-d}\,, \qquad (4.19)\end{aligned}$$

due to the choice $N\lambda^{d-2}=1$.

We claim that there exists a family of points $(a_i)_{1\le i\le N}$ such that the last contribution in the r.h.s of (4.19) is uniformly bounded with respect to N. The argumentation goes as follows. First we may change the reference norm, i.e. we can replace the euclidean norm $|\cdot|_2$ in $\mathbb{R}^d$ with the supremum norm $|\cdot|_\infty$, up to some constant. Then, we distribute N points on the regular grid $N^{-1/d}\cdot\mathbb{Z}^d$ inside the hypercube $[-1/2,1/2]^d$. Next, we observe that for any index i and any integer $k < \frac{1}{2}N^{1/d}$ there are at most Ck^{d-1} indices j such that $|a_i - a_j|_\infty = N^{-1/d}k$, where C is a constant depending only on the dimension. As a matter of fact, after rescaling space by a factor $N^{1/d}$, those points are regularly distributed on a sphere of radius k. Finally, we have the following estimates as $N \to +\infty$: $\lambda \ll N^{-1/d}$ and consequently $D^\lambda(i,j) \sim |a_i - a_j|_2$. To conclude, we can estimate the last contribution in the r.h.s of (4.19) as follow

$$\begin{aligned}\frac{1}{N^2}\sum_{i\neq j}(D^\lambda(i,j))^{2-d} &\leq C\frac{1}{N^2}\,N\sum_{k=1}^{\lfloor 2^{-1}N^{1/d}\rfloor} k^{d-1}\left(N^{-1/d}k\right)^{2-d}\\ &\leq C\frac{1}{N}\,N^{(d-2)/d}\,\frac{1}{4}\,N^{2/d} = \frac{1}{4}\,C\,,\end{aligned}$$

Therefore the interaction potential is bounded from below, while the entropy (4.18) is decreasing towards $-\infty$ as $N \to +\infty$. Moreover, it is always possible to scale appropriately the location of the family (here inside $[-1/2, 1/2]^d$ for the sake of reference) to ensure that the right inequality in (4.16) is satisfied.

To conclude, we discuss evidence showing that criterion (1.8) is more generic than criterion (4.5). The argumentation is twofold. First, the example of an initial data n_0 satisfying (4.5) but not (1.8) is much more involved than the converse, namely a superposition of numerous approximations of the identity with careful scaling.

Second, we have designed a numerical scheme preserving the energy structure of the system, following [38, 10]. For technical reasons, we have transposed the PE system into a one-dimensional equation which shares similar features. Instead of working directly with the density n, we have approximated the inverse distribution function $X(t,m) = \inf\left\{x \in \mathbb{R}\;\middle|\; \int_{-\infty}^x n(t,y)\,dy \geq m\right\}$. To sum up, the numerical scheme we have developped is the euclidean gradient flow $\dot{X} = -\nabla\mathcal{G}_N[X]$, where $\mathcal{G}_N$ is given by

$$\begin{aligned}\mathcal{G}_N[X](t) = -\frac{1}{N}\sum_{i=1}^{N-1}\log\left(\frac{X_{i+1}(t) - X_i(t)}{h}\right)&\\ - \frac{1}{8\pi N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}|X_j(t) - X_i(t)|^{-1}\,.&\end{aligned}$$

The discrete energy $\mathcal{G}_N$ is the finite-difference discretisation of the energy (1.6), expressed after the change of variables $x = X(t,m)$ and $n(t,x)dx = dm$, [27]. In this setting, the functions $(X_i(t))_{1\leq i\leq N}$ can be interpreted as the positions of N deterministic particles in interaction. It is shown in [38, 10] that this approach is well-adapted to the energy structure of the system. The case $N = 3$ (discretisation with only 3 particles) is very instructive. We can easily investigate the full dynamics of the three particles system. Indeed, we have checked that the corresponding criterion (1.8) strictly contains the other one (4.5), and also that the criteria (1.8) (blow-up) and (1.7) (global existence) are clearly disconnected.

5 Concentration phenomenon for the PP system

In this section we shall prove Theorem 1.3, where $\alpha = 0$ only for the sake of clarity. The key strategy giving the concentration result is the perturbation of the PP system into the PE system. Doing that, one is naturally induced to analyze the evolution of the corrected energy $\mathcal{F}[n,c]$ defined in (1.9). The delicate and original points of the proof consist in the careful control of $\|\partial_t c\|_{L^2}$ and of the growth of the second moment $I(t)$ (no larger than $t\log t$), both by the dissipation of the free energy (1.5). To begin we need the two auxiliary Lemma below. The first one is an adaptation of Lemma 2.2 in [16] in a simpler context. Therefore, we skip the proof.

Lemma 5.1 (Free energy minimization)**.** *Let $d \geq 3$. Let n be any nonnegative function in $(L^1 \cap L^{2d/(d+2)})(\mathbb{R}^d)$, such that $\int_{\mathbb{R}^d} n(x)\log n(x)\,dx < \infty$. Let $\overline{c} := E_d * n$. Then, the energy functional defined in* (1.4) *satisfies*

$$\mathcal{E}[n,c] \geq \mathcal{E}[n,\overline{c}] = \int_{\mathbb{R}^d} n(x)\log n(x)\,dx - \frac{1}{2}\int_{\mathbb{R}^d} n(x)\,\overline{c}(x)\,dx\ ,$$

for any c such that $nc \in L^1(\mathbb{R}^d)$ and $\nabla c \in L^2(\mathbb{R}^d)$.

Lemma 5.2 (Corrected energy lower bound)**.** *Let n be any nonnegative function in $(L^1 \cap L^{\frac{d}{2}})(\mathbb{R}^d)$ with finite second moment. Let c be such that $nc \in L^1(\mathbb{R}^d)$ and $\nabla c \in L^2(\mathbb{R}^d)$. The following lower bound for the corrected energy $\mathcal{F}[n,c]$ holds true, where the constant $B(d,M)$ is defined in* (4.9).

$$d(d-2)M\mathcal{F}[n,c] + B(d,M) + \frac{1}{|\mathbb{S}^{d-1}|}M\,C_{HLS}(d,d-2)\|n\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \geq 2dM\,. \tag{5.1}$$

Proof. This is a direct consequence of Lemma 5.1 and of inequalities (2.4) and (2.6). $\square$

Proof of Theorem 1.3. We proceed in several steps to conclude with a contradiction argument.
First step: evolution of $\mathcal{F}[n,c]$. We express c as: $c = E_d * (n - \varepsilon\,\partial_t c)$. Thus, the gradient of c can be written as follows:

$$\nabla c(x,t) = -\frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d}\,(n(y,t) - \varepsilon\,\partial_t c(y,t))\,dy\,.$$

Introducing this representation into the evolution equation (4.2) for the second moment $I(t)$, we get after symmetrization

$$\frac{d}{dt}I(t) = 2dM - (d-2)\int_{\mathbb{R}^d} n(t)\overline{c}(t) + \frac{2\varepsilon}{|\mathbb{S}^{d-1}|}\iint_{\mathbb{R}^d\times\mathbb{R}^d} n(x,t)\,\frac{x\cdot(x-y)}{|x-y|^d}\,\partial_t c(y,t)dxdy \tag{5.2}$$

where $\bar{c} = E_d * n$. In order to control the second integral term in the r.h.s. of (5.2), we apply the Hardy-Littlewod-Sobolev inequality [43],

$$
\begin{aligned}
\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x| n(x,t)\,\frac{|\partial_t c(y,t)|}{|x-y|^{d-1}}\,dxdy &\le C(d)\|x\,n(t,\cdot)\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}\,\|\partial_t c(t)\|_{L^2(\mathbb{R}^d)}\\
&\le C(d)\,I^{\frac12}(t)\|n(t)\|^{\frac12}_{L^{\frac d2}(\mathbb{R}^d)}\|\partial_t c(t)\|_{L^2(\mathbb{R}^d)}\\
&\le C(d)\left[\delta\,\|n(t)\|_{L^{\frac d2}(\mathbb{R}^d)}+\delta^{-1}I(t)\|\partial_t c(t)\|^2_{L^2(\mathbb{R}^d)}\right],
\end{aligned}
\tag{5.3}
$$

where the constant $C(d)$ above may change from line to line, the last being the one appearing in (1.11). Plugging (5.3) into (5.2), we obtain our first main estimate (the equivalent of (4.3) for $\varepsilon > 0$)

$$
\begin{aligned}
\frac{d}{dt}I(t) \le 2dM-(d-2)\int_{\mathbb{R}^d} n(t)\bar{c}(t)+\varepsilon\,\delta\,C(d)\,\|n(t)\|_{L^{\frac d2}(\mathbb{R}^d)}&\\
+\,\varepsilon\,\delta^{-1}C(d)\,I(t)\|\partial_t c(t)\|^2_{L^2(\mathbb{R}^d)}&\,.
\end{aligned}
$$

Next, we use Lemma 5.1 and the energy dissipation equation (1.5), to have

$$
\begin{aligned}
\frac{d}{dt}I(t) \le 2dM+2(d-2)\left[\mathcal{E}[n,c](t)-\int_{\mathbb{R}^d} n(t)\log n(t)\right]&\\
+\,\varepsilon\delta\,C(d)\,\|n(t)\|_{L^{\frac d2}(\mathbb{R}^d)}-\delta^{-1}C(d)\,I(t)\,\frac{d}{dt}\mathcal{E}[n,c](t)&\,.
\end{aligned}
$$

Choosing $\delta = \frac{dM}{2}\,C(d)$, the entropy lower bound (2.4) gives us

$$
\begin{aligned}
\frac{d}{dt}I(t) \le\;& 2dM+2(d-2)\,\mathcal{E}[n,c](t)\\
&+2(d-2)\left[\frac{dM}{2}\log I+\frac{dM}{2}-M\log M-\frac{dM}{2}\log\left(\frac{dM}{2\pi}\right)\right]\\
&+\frac{\varepsilon}{2}\,dM\,C^2(d)\,\|n(t)\|_{L^{\frac d2}(\mathbb{R}^d)}-\frac{2}{dM}\,I(t)\,\frac{d}{dt}\mathcal{E}[n,c](t)\,,
\end{aligned}
$$

i.e., after rearranging the terms, we get the following differential inequality for the corrected energy (1.9):

$$
I(t)\,\frac{d}{dt}\mathcal{F}[n,c](t) \le d(d-2)M\,\mathcal{F}[n,c](t)+B(d,M)+\frac{\varepsilon}{2}\,dM\,C^2(d)\,\|n(t)\|_{L^{\frac d2}(\mathbb{R}^d)}\,.
\tag{5.4}
$$

Second step: control of the growth of $I(t)$. Let us prove that the second moment does not increase asymptotically faster than $2dM\,t\log t$ as long as $\|n(t)\|_{L^{\frac d2}(\mathbb{R}^d)}$ remains bounded from above. Indeed,

$$
\begin{aligned}
\frac{d}{dt}I(t) &= -2\int_{\mathbb{R}^d} n(x,t)\;x\cdot\nabla(\log n(x,t)-c(x,t))\,dx\\
&\le 2\left(\int_{\mathbb{R}^d}|x|^2 n(x,t)\,dx\right)^{1/2}\left(\int_{\mathbb{R}^d} n(x,t)|\nabla(\log n(x,t)-c(x,t))|^2\,dx\right)^{1/2}.
\end{aligned}
$$

Therefore,

$$\frac{d}{dt} I^{1/2}(t) \leq \left(\int_{\mathbb{R}^d} n(x,t) |\nabla (\log n(x,t) - c(x,t))|^2 \, dx \right)^{1/2} .$$

Integrating the above inequality over $(0,t)$ we get

$$I(t) \leq 2I(0) + 2\,t \int_0^t \int_{\mathbb{R}^d} n(x,s) |\nabla (\log n(x,s) - c(x,s))|^2 \, dx ds \,.$$

Using again the energy dissipation equation (1.5), we derive the following pointwise estimate for $I(t)$ with respect to the corrected energy $\mathcal{F}[n,c](t)$

$$\begin{aligned} I(t) &\leq 2I(0) + 2\,t\,(\mathcal{E}[n_0,c_0] - \mathcal{E}[n,c](t)) \\ &= 2I(0) + 2\,t\,\mathcal{E}[n_0,c_0] - dMt\,\mathcal{F}[n,c](t) + dMt\,\log I(t)\,. \end{aligned}$$

Finally, being $dMt\,\log I \leq \frac{1}{2}\,I + dMt\,\log(2dM\,t)$, we obtain

$$I(t) \leq 4I(0) + 4\,t\,\mathcal{E}[n_0,c_0] - 2dMt\,\mathcal{F}[n,c](t) + 2dMt\,\log(2dM\,t)\,,$$

i.e. the claimed behaviour for $I(t)$, thanks to the lower bound (5.1)

$$I(t) \leq 4I(0) + 4t\,\mathcal{E}[n_0,c_0] + 2t \left[\frac{B(d,M)}{d-2} + \mu_d\, M\, C_{HLS}(d,d-2) \|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \right] + 2dMt\,\log(2dM\,t)\,. \quad (5.5)$$

Third step: contradiction. Let us observe that the concentration condition (1.10) reads equivalently as

$$d(d-2)M\mathcal{F}[n_0,c_0] + B(d,M) + d(d-2)M\,\varepsilon^{\gamma} < 0\,. \quad (5.6)$$

Comparing the master equation (5.4) and (5.6), we claim that we can not have uniformly in time

$$\frac{\varepsilon}{2}\, dM\, C^2(d)\, \|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} < d(d-2)M\,\varepsilon^{\gamma}\ . \quad (5.7)$$

Indeed, if n_0 does not satisfy (5.7), the conclusion is obvious. On the other hand, if n_0 satisfies (5.7), we deduce from (5.4) that the corrected energy is initially decreasing. Next, if (5.7) holds true uniformly in time, there exists $\delta > 0$ such that $I(t)\frac{d}{dt}\mathcal{F}[n,c](t) < -\delta$ for $t > 0$. Combining (5.7) and (5.5), we get

$$\frac{d}{dt}\mathcal{F}[n,c](t) \leq \frac{-\delta}{4I(0) + C(\varepsilon,d,M,\mathcal{E}[n_0,c_0])\,t + 2dM\,t\log t}\,. \quad (5.8)$$

The right-hand-side of (5.8) is not integrable at infinity, whereas $\mathcal{F}[n,c](t)$ is bounded from below as soon as $\|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$ is bounded from above. We have obtained a contradiction which concludes the proof. □

Appendix

We provide in this Section the details omitted in the proof of Proposition 3.1. This completes also the proof of Theorem 1.1

Local in time hyper-contractivity property

With $a > \frac{d}{2}$ and $T_a > 0$ defined in (3.5), let $\delta \in (0, T_a)$ be arbitrarily small. Owing to the local in time boundedness of the L^a-norm of n obtained in the first step, there exists a modulus of "$\frac{d}{2}$-equintegrability" $\omega(K; T_a - \delta)$ such that for $K \geq 1$ it holds

$$\sup_{0 \leq t \leq (T_a - \delta)} \|(n-K)_+(t)\|^{\frac{d}{2}}_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq \omega(K; T_a - \delta) \quad \text{and} \quad \lim_{K \to +\infty} \omega(K; T_a - \delta) = 0 \,.$$

Indeed, for $t \in [0, T_a - \delta]$, being $h_a(t)$ defined in (3.6) increasing, we have

$$\|(n-K)_+(t)\|^{\frac{d}{2}}_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq \frac{1}{K^{a-\frac{d}{2}}} \int_{\mathbb{R}^d} n^a(t) \leq \frac{h_a(T_a - \delta)}{K^{a-\frac{d}{2}}} \int_{\mathbb{R}^d} n_0^a =: \omega(K; T_a - \delta) \,. \tag{5.9}$$

Next, it holds, for any fixed $p > \max\{2; a\}$,

$$\frac{d}{dt} \int_{\mathbb{R}^d} (n-K)_+^p \leq -4 \frac{(p-1)}{p} \int_{\mathbb{R}^d} |\nabla (n-K)_+^{\frac{p}{2}}|^2 + 4(p-1) \int_{\mathbb{R}^d} (n-K)_+^{p+1} + 4K^p M. \tag{5.10}$$

Using the inequality

$$\int_{\mathbb{R}^d} v^{p+1} \leq \|v\|^p_{L^{\frac{pd}{d-2}}(\mathbb{R}^d)} \, \|v\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq C_S^2(d) \|\nabla v^{\frac{p}{2}}\|^2_{L^2(\mathbb{R}^d)} \, \|v\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \,, \tag{5.11}$$

where $p \geq \max\{\frac{d}{2} - 1\,;\, 1\}$, and (5.9) to estimate $\int_{\mathbb{R}^d} (n-K)_+^{p+1}$, (5.10) becomes

$$\frac{d}{dt} \int_{\mathbb{R}^d} (n-K)_+^p \leq -4\,(p-1) \left[\frac{1}{p\, C_S^2(d) \omega^{\frac{2}{d}}(K; T_a - \delta)} - 1 \right] \int_{\mathbb{R}^d} (n-K)_+^{p+1} + 4K^p M \,.$$

Taking K sufficiently large, so that $\eta = \left[p^{-1} C_S^{-2}(d) \omega^{-\frac{2}{d}}(K; T_a - \delta) - 1 \right] > 0$,

$$\frac{d}{dt} \int_{\mathbb{R}^d} (n-K)_+^p \leq -4(p-1)\, \eta \, M^{-\frac{1}{p-1}} \left(\int_{\mathbb{R}^d} (n-K)_+^p \right)^{\frac{p}{p-1}} + 4K^p M \,. \tag{5.12}$$

We are finally able to prove that there exists a positive finite constant C, not depending on $\int_{\mathbb{R}^d} (n_0 - K)_+^p$, such that for $p > \max\{2; a\}$ and $t \in (0, T_a - \delta]$

$$\int_{\mathbb{R}^d} (n-K)_+^p(t) \leq \frac{C}{t^{\;p-1}} \,,$$

simply by comparison of positive solutions of (5.12) with positive solutions of the differential equation $u'(t) + 4(p-1)\, \eta \, M^{-\frac{1}{p-1}} \, u^{\frac{p}{p-1}}(t) = 4K^p M$. Consequently and as usual, the hypercontractivity estimate

$$\int_{\mathbb{R}^d} n^p(t) \le C(1 + t^{1-p}) \,, \qquad \text{a.e.} \;\; t \in (0, T_a)\,, \tag{5.13}$$

holds true for any $p > \max\{2; a\}$. For $p \in (a, \max\{2; a\}]$ it follows by interpolation.

From the previous estimates, the chemical c and ∇c are well defined and by the weak Young inequality [43], (see also Lemma 2.3),

$$\|c(t)\|_{L^q(\mathbb{R}^d)} \le C(d, \alpha)\, \|n(t)\|_{L^p(\mathbb{R}^d)}\,, \quad \text{a.e. } t \in (0, T_a)\;,$$

for $q = \frac{pd}{d-2p}$ with $p \in (1, \frac{d}{2})$, while

$$\|\nabla c(t)\|_{L^r(\mathbb{R}^d)} \le C(d, \alpha)\, \|n(t)\|_{L^p(\mathbb{R}^d)}\,, \quad \text{a.e. } t \in (0, T_a)\,, \tag{5.14}$$

for $r = \frac{pd}{d-p}$ with $p \in (1, d)$.

Entropy, potential, energy and energy dissipation estimates

We will show here how a local in time control on the $L^{\frac{d}{2}}$ norm of n together with a control on the decay of n_0 as $|x| \to \infty$, give local in time estimates on the entropy, the potential, the energy and the dissipation of the energy.

For the potential, from (1.1), the expansion of B_d^α in Lemma 2.3 and the potential confinement Lemma 2.2, we have easily

$$0 \le \int_{\mathbb{R}^d} n(t)\, c(t) \le \mu_d\, C_{HLS}(d, d-2) M \|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}\,. \tag{5.15}$$

Next, for the positive contribution of the inital entropy $\int_{\mathbb{R}^d} n_0 \log n_0$ it holds

$$0 \le \int_{\mathbb{R}^d} (n_0 \log n_0)_+ \le \|n_0\|^{\frac{d}{2}}_{L^{\frac{d}{2}}(\mathbb{R}^d)}\,. \tag{5.16}$$

Moreover, setting $v = n_0\, 1\!\!1_{\{n_0 \le 1\}}$ and $m = \int_{\mathbb{R}^d} v$, the Jensen inequality gives us

$$\int_{\mathbb{R}^d} (v \log v + v\psi) = \int_{\mathbb{R}^d} v \log\left(\frac{v}{e^{-\psi}}\right) \ge m \log m - m \log(\|e^{-\psi}\|_{L^1(\mathbb{R}^d)}),$$

which implies

$$0 \le \int_{\mathbb{R}^d} (n_0 \log n_0)_- \le C(m, \|e^{-\psi}\|_{L^1(\mathbb{R}^d)}) + \int_{\mathbb{R}^d} n_0 \psi\;. \tag{5.17}$$

Estimates, (5.15), (5.16) and (5.17) give us for the initial energy (1.6) that

$$-C(m, \|e^{-\psi}\|_{L^1(\mathbb{R}^d)}) - \int_{\mathbb{R}^d} n_0 \psi - C(M, d) \|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \le \mathcal{E}[n_0] \le \|n_0\|^{\frac{d}{2}}_{L^{\frac{d}{2}}(\mathbb{R}^d)}\,.$$

Now, let us extend the above estimates locally in time. Estimate (5.16) is obviously true for $n(t)$. Concerning the weighted L^1-norm of n we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} n\psi = -\int_{\mathbb{R}^d} n\nabla\psi\cdot\nabla(\log n - c) \le \frac{1}{4\delta}\int_{\mathbb{R}^d} \left|\nabla\psi\right|^2 n + \delta\int_{\mathbb{R}^d} n|\nabla(\log n - c)|^2 \,,$$

with arbitrary $\delta > 0$ to be choosen later. Hence, it follows that

$$\int_{\mathbb{R}^d} n(t)\psi \le \int_{\mathbb{R}^d} n_0\psi + \frac{1}{4\delta}\|\nabla\psi\|^2_{L^\infty(\mathbb{R}^d)} M\,t + \delta\int_0^t\int_{\mathbb{R}^d} n|\nabla(\log n - c)|^2 \,. \tag{5.18}$$

It remains to estimate the energy dissipation from (1.5) with $\varepsilon = 0$. Using (5.15) and (5.17), we obtain

$$\begin{aligned}\textstyle\int_0^t\int_{\mathbb{R}^d} \quad & n(s)|\nabla(\log n(s) - c(s))|^2 \; dx \; ds = \mathcal{E}[n_0] - \mathcal{E}[n](t) \\ & \le \|n_0\|^{\frac{d}{2}}_{L^{\frac{d}{2}}(\mathbb{R}^d)} + \textstyle\int_{\mathbb{R}^d}(n(t)\log n(t))_- + \frac{1}{2}\int_{\mathbb{R}^d} n(t)c(t) \\ & \le \|n_0\|^{\frac{d}{2}}_{L^{\frac{d}{2}}(\mathbb{R}^d)} + C(m, \|e^{-\psi}\|_{L^1(\mathbb{R}^d)}) + \textstyle\int_{\mathbb{R}^d} n(t)\psi + C(M,d)\|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \,.\end{aligned} \tag{5.19}$$

Plugging now (5.18) with any $\delta \in (0,1)$ into (5.19), we finally get

$$(1-\delta)\int_0^t\int_{\mathbb{R}^d} n|\nabla(\log n - c)|^2 \le C(n_0,\psi,M)(1+t) + C(M,d)\|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \,.$$

Consequently, from (5.18), the same estimate holds for the weighted L^1-norm of n, i.e. $\int_{\mathbb{R}^d} n(t)\psi$, and this gives us the following control on the energy

$$-C(n_0,\psi,M)(1+t) - C(M,d)\|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \le \mathcal{E}[n](t) \le \|n(t)\|^{\frac{d}{2}}_{L^{\frac{d}{2}}(\mathbb{R}^d)} \,, \tag{5.20}$$

and exactly the same control on the entropy.

Regularisation procedure and conclusion

To prove Proposition 3.1 from the previous a-priori estimates, we follow a quite standard procedure, used in the context of Keller-Segel system for example in [12, 16, 21, 22]. However, since its application can be delicate, we give here a quite rapid sketch of the proof.

Let us consider the regularized problem

$$\partial_t n^\sigma = \Delta n^\sigma - \nabla\cdot(n^\sigma\nabla c^{\,\sigma})\,, \tag{5.21}$$

where $c^{\,\sigma}$ is given by

$$c^{\,\sigma}(x,t) = \begin{cases} (E_d * n^\sigma(t) * \rho^\sigma)(x)\,, & \alpha = 0\,, \\ (B_d^\alpha * n^\sigma(t) * \rho^\sigma)(x)\,, & \alpha > 0\,, \end{cases} \tag{5.22}$$

and ρ^σ is some sequence of smooth positive mollifiers with $\|\rho^\sigma\|_{L^1(\mathbb{R}^d)} = 1$.

Problem (5.21)-(5.22) with regularized initial condition $n_0^\sigma = n_0 * \rho^\sigma$ has a nonnegative smooth solution as follows by the Schauder's fixed-point theorem. Moreover, the solution (n^σ, c^σ) satisfies all the a priori estimates given in the previous sections. Indeed, one has essentially to check that the ODE for the L^p norm of n holds true at least as an inequality, so that the fundamental estimates on the L^{p+1} norm of n^σ can be applied. This is the case since

$$-\int_{\mathbb{R}^d} (n^\sigma)^p \, \Delta c^\sigma \le \int_{\mathbb{R}^d} (n^\sigma)^p \, (n^\sigma * \rho^\sigma) \le \int_{\mathbb{R}^d} (n^\sigma)^{p+1} \, .$$

Concerning the compactness of the sequence $\{n^\sigma\}$, we are intended to use the Aubin compactness Lemma in [1]. Therefore, we choose $B = L^2(\mathbb{R}^d)$, $X = H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, \psi(x)^{\frac{1}{2}} dx)$ compactly imbedded in B and $Y = H^{-1}(\mathbb{R}^d)$ so that $B \subset Y$. Using the previous a priori estimates, we prove below that $\{n^\sigma\}$ is bounded in $L^2((\delta, T_a - \delta); X)$ uniformly in σ, where $\delta \in (0, T_a)$ is arbitrarily small and T_a is defined in (3.6). The boundedness of $\{\partial_t \, n^\sigma\}$ in $L^2((\delta, T_a - \delta); Y)$ uniformly in σ then follows. Consequently, $\{n^\sigma\}$ is relatively compact in $L^2((\delta, T_a - \delta); B)$ and the proof is complete.

For the sake of simplicity, we omit the index σ in the sequel. First of all, let us observe that $n \in L^\infty((\delta, T_a - \delta)\,; L^2(\mathbb{R}^d))$ if $d = 3$ and $n \in L^\infty((0, T_a - \delta)\,; L^2(\mathbb{R}^d))$ if $d \ge 4$, as it follows by (5.13) and by (3.5) respectively, being $a > \frac{d}{2}$.

Next, from (5.14) with $p = \frac{d}{2}$, we have that $|\nabla c| \in L^\infty((0, T_a - \delta)\,; L^d(\mathbb{R}^d))$, for any $d \ge 3$. Consequently, since by the Hölder's inequality,

$$\|(n \, \nabla c)(t)\|_{L^2(\mathbb{R}^d)} \le \|n(t)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \|\nabla c(t)\|_{L^d(\mathbb{R}^d)} \, ,$$

using again (5.13), we obtain that $n|\nabla c| \in L^\infty((\delta, T_a - \delta)\,; L^2(\mathbb{R}^d))$ for $d \ge 3$.

Finally, multiplying the equation on n in (KS) against n, integrating over $\mathbb{R}^d$ and then over $(\delta, T_a - \delta)$, one easily obtain

$$\int_\delta^{T_a - \delta} \|\nabla n(s)\|^2_{L^2(\mathbb{R}^d)} \, ds \le \|n(\delta)\|^2_{L^2(\mathbb{R}^d)} + \int_\delta^{T_a - \delta} \|(n \, \nabla c)(s)\|^2_{L^2(\mathbb{R}^d)} \, ds \, ,$$

i.e. $|\nabla n| \in L^2((\delta, T_a - \delta)\,; L^2(\mathbb{R}^d))$ for $d \ge 3$.

To estimate the $L^2(\mathbb{R}^d, \psi(x)^{\frac{1}{2}} dx)$ norm of n, we use the computation

$$\int_{\mathbb{R}^d} \psi^{\frac{1}{2}} \, n^2(t) \le \left(\int_{\mathbb{R}^d} \psi \, n(t) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} n^3(t) \right)^{\frac{1}{2}} \, ,$$

and the previous estimates. Therefore, $n \in L^\infty((\delta, T_a - \delta)\,; L^2(\mathbb{R}^d, \psi(x)^{\frac{1}{2}} dx))$ for $d \ge 3$ and the proof is complete.

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