

A one-dimensional Keller-Segel equation with a drift issued from the boundary

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Abstract

We investigate in this note the dynamics of a one-dimensional Keller-Segel type model on the half-line. On the contrary to the classical configuration, the chemical production term is located on the boundary. We prove, under suitable assumptions, the following dichotomy which is reminiscent of the two-dimensional Keller-Segel system. Solutions are global if the mass is below the critical mass, they blow-up in finite time above the critical mass, and they converge to some equilibrium at the critical mass. Entropy techniques are presented which aim at providing quantitative convergence results for the subcritical case. This note is completed with a brief introduction to a more realistic model (still one-dimensional).

Résumé

Nous étudions dans cette note la dynamique d'un modèle unidimensionnel de type Keller-Segel posé sur une demi-droite. Dans le cas présent, la production du signal chimique est localisée sur le bord, au lieu d'être répartie à l'intérieur du domaine comme dans le cas classique. On démontre, sous des hypothèses convenables, la dichotomie suivante qui rappelle le système de Keller-Segel en dimension deux d'espace. Les solutions sont globales si la masse est sous-critique, elles explosent en temps fini si la masse dépasse la masse critique. Enfin, les solutions convergent vers un état d'équilibre lorsque la masse est égale à la valeur critique. Des méthodes d'entropie sont développées, dans le but d'obtenir des résultats de convergence quantitatifs. Cette note est enrichie d'une brève introduction à un modèle plus réaliste (à nouveau unidimensionnel).

Version française abrégée

Dans cette note nous allons étudier le comportement mathématique en dimension un de l'équation aux dérivées partielles suivante :

$$\partial_t n(t, x) = \partial_{xx} n(t, x) + n(t, 0) \partial_x n(t, x), \quad t > 0, x \in (0, +\infty), \quad (1)$$

avec la condition initiale : $n(t = 0, x) = n_0(x) \geq 0$. Nous imposons au bord une condition de flux nul : $\partial_x n(t, 0) + n(t, 0)^2 = 0$, de sorte que la masse est conservée au cours du temps (au moins formellement) :

$$\int_{x>0} n(t, x) dx = \int_{x>0} n_0(x) dx = M. \quad (2)$$

Ce modèle a été proposé dans [15] pour décrire synthétiquement la polarisation des cellules de levure. Une caractéristique intéressante de (1) réside dans le fait que la solution peut devenir non bornée en temps fini. Dans cette note nous allons montrer l'alternative suivante :

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Théorème 1 (Existence globale vs. explosion) *Supposons que n_0 est continue sur $[0, +\infty)$ et que $n_0 \in L^1_+((1+x^2)dx)$. Si $M \leq 1$ alors il existe une solution faible de (1) qui est globale en temps. Au contraire si $M > 1$, en supposant en outre que n_0 est décroissante, alors toute solution forte explose en temps fini.*

Nous annonçons également les résultats suivants concernant le comportement asymptotique de la solution lorsque $M \leq 1$:

Théorème 2 (Comportement asymptotique) *Dans le cas critique $M = 1$, il existe une famille d'états stationnaires pour (1) paramétrée par $\alpha > 0$. La solution converge (au sens de l'entropie relative (14)) vers l'équilibre tel que $\alpha^{-1} = \int_{x>0} xn_0(x) dx$. Dans le cas sous-critique $M < 1$, la solution décroît vers zéro, et converge (au sens de l'entropie relative) vers un unique profil auto-similaire.*

Enfin, nous nous intéressons à l'étude d'un modèle plus réaliste, qui prend en compte l'échange entre des particules libres de densité $n(t, x)$, $x \in (0, +\infty)$, et des particules fixées au bord $\{x = 0\}$, de concentration $\mu(t)$, qui créent le potentiel attractif :

$$\begin{cases} \partial_t n(t, x) = \partial_{xx} n(t, x) + \mu(t) \partial_x n(t, x), & t > 0, x \in (0, +\infty), \\ \mu'(t) = n(t, 0) - \mu(t), \end{cases}$$

avec la condition de flux au bord : $\partial_x n(t, 0) + \mu(t)n(t, 0) = \mu'(t)$ (voir [9, 15] pour plus de détails liés à la modélisation).

Théorème 3 *Avec les hypothèses des théorèmes précédents, et dans le cas sur-critique $M > 1$, $\mu(t)$ converge vers $\bar{\mu} = M - 1$ et la densité $n(t, x)$ converge en entropie relative vers $h(x) = \bar{\mu} \exp(-\bar{\mu}x)$.*

English Version

In this note we shall study the mathematical behavior of the following one dimensional partial differential equation:

$$\partial_t n(t, x) = \partial_{xx} n(t, x) + n(t, 0) \partial_x n(t, x), \quad t > 0, x \in (0, +\infty), \quad (3)$$

together with the initial condition: $n(t = 0, x) = n_0(x) \geq 0$. We impose a zero-flux boundary condition for the density n ,

$$\partial_x n(t, 0) + n(t, 0)^2 = 0. \quad (4)$$

Notice that (4) and $n_0 \in L^1_+$ guarantees nonnegative solutions $n(t, x) \geq 0$ and mass conservation (at least formally):

$$\int_{x>0} n(t, x) dx = \int_{x>0} n_0(x) dx = M. \quad (5)$$

This model has been proposed in [15] to describe basically the polarisation of cells. In short the cell is viewed as the half-line $(0, +\infty)$ and $n(t, x)$ denotes the density of some cellular markers which are transported by the intracellular cytoskeleton (actin or microtubules polymers). The nonlinear coupling is due to the growth of new polymers which is linked to the density of markers at the membrane $\{x = 0\}$.

The interesting feature of (3)–(4) is that the solution may become unbounded in finite time. Such a behavior is called *blow-up in finite time*. To be more precise, we define a weak solution to (3) on $(0, T)$ as follows: $n(\cdot, 0) \in L^2(0, T)$, $n \in L^\infty(0, T; L^1(\mathbb{R}_+))$, $\partial_x \sqrt{n} \in L^2(0, T; L^2(\mathbb{R}_+))$, and n verifies (3) in the sense of distributions. Note that this definition ensures that $n(t, \cdot)$ is continuous almost every time (11), therefore the trace value $n(t, 0)$ is well defined. We say that the solution blows-up if $\int_{x>0} n(t, x) (\log n(t, x))_+ dx$ becomes unbounded.

In this note we shall prove the following simple alternative:

Theorem 1 *Assume n_0 is continuous on $[0, +\infty)$ and $n_0 \in L^1_+((1+x^2)dx)$. If $M \leq 1$ there exists a weak solution of (3)–(4) which is global in time. On the contrary if $M > 1$ and n_0 is nonincreasing, any strong solution blows-up in finite time.*

We do not address the uniqueness issue here. Concerning global existence we only give very sketchy arguments. A complete proof will be given in [9]

This dichotomy between global existence and blow-up is reminiscent of the two-dimensional Keller-Segel (KS) system (see [6] and references therein). It describes macroscopically a population of diffusive cells which attract each other through a diffusive chemical signal given by the Poisson equation: $-\Delta c(t, x) = n(t, x)$ with homogeneous Neumann boundary conditions [21, 20, 22, 19]. On the other hand the chemical field in (3) would be solution of the Laplace equation with inhomogeneous Neumann boundary conditions: $-\partial_x c(t, 0) = n(t, 0)$ (production of the signal is located on the boundary). Although the Keller-Segel cannot exhibit blowing-up solutions in one dimension of space, it is indeed the case for (3). As a conclusion, (3) appears to be as singular as the two-dimensional Keller-Segel system. Note that there exist other ways to mimick the singular behaviour of the two-dimensional KS in one dimension [3, 10, 11].

Remark 2 *It would be possible to weaken the assumptions on the regularity of the initial data n_0 (basically $\int_{x>0} n_0(x) |\log n_0(x)| dx < +\infty$ instead of n_0 continuous) by using strong regularizing effects of the laplacian (at least in the subcritical case $M < 1$) but this is beyond the scope of this note.*

Remark 3 *There is a strong connection between the equation under interest here (3) and the one-dimensional Stefan problem. The later writes [16]:*

$$\begin{cases} \partial_t u(t, z) = \partial_{zz} u(t, z), & t > 0, z \in (-\infty, s(t)), \\ \lim_{z \rightarrow -\infty} \partial_z u(t, z) = 0, & u(t, s(t)) = 0, \quad \partial_z u(t, s(t)) = -s'(t). \end{cases} \quad (6)$$

The temperature is initially nonnegative: $u(0, z) = u_0(z) \geq 0$. By performing the following change of variables: $\phi(t, x) = -u(t, s(t) - x)$, we get an equation which is linked to (3) by $n(t, x) = \partial_x \phi(t, x)$. This connection provides some insights concerning the possible continuation of solutions after blow-up [16]. This question has raised a lot of interest in the past recent years [17, 24, 25, 14]. It is postulated in [16] that the one-dimensional Stefan problem is generically non continuable after the blow-up time.

1 The critical mass phenomenon

Blow-up for $M > 1$. To prove that solutions blow-up in finite time, we show that the first momentum of $n(t, x)$ cannot remain positive for all time. This technique was first used by Nagai [22], then by many authors in various contexts (see [2, 3, 12, 13, 11] for instance). Other strategies have been used to prove the existence of blowing-up solutions (either constructive by Herrero and Velázquez [17] or undirect [18]), however up to date this trick is the only way to provide explicit criterion and appears to be quite robust to variations around Keller-Segel [4, 8].

First, the assumption that n_0 is a nonincreasing function guarantees that $n(t, \cdot)$ is also a non-increasing function for any time $t > 0$ due to the maximum principle (notice that the derivative $v(t, x) = \partial_x n(t, x)$ satisfies a parabolic type equation, is initially nonpositive, and is nonpositive on the boundary due to (4)). Therefore $-\partial_x n(t, x)/n(t, 0)$ is a probability density at any time $t > 0$. We deduce from Jensen's inequality the following interpolation estimate:

$$\begin{aligned} \left(\int_{x>0} x \frac{-\partial_x n(t, x)}{n(t, 0)} dx \right)^2 &\leq \int_{x>0} x^2 \frac{-\partial_x n(t, x)}{n(t, 0)} dx, \\ M^2 &\leq 2n(t, 0) \int_{x>0} xn(t, x) dx. \end{aligned}$$

Secondly introduce the first momentum $\mathbf{J}(t) = \int_{x>0} xn(t, x) dx$. We have for $M > 1$:

$$\begin{aligned} \frac{d\mathbf{J}(t)}{dt} &= n(t, 0) - Mn(t, 0) \leq \frac{M^2}{2\mathbf{J}(t)}(1 - M), \\ \frac{d\mathbf{J}(t)^2}{dt} &\leq M^2(1 - M). \end{aligned} \quad (7)$$

Therefore blow-up of the solution occurs in finite time if $M > 1$.

Global existence for $M < 1$. Global existence results for Keller-Segel type systems have been initiated by Jäger and Luckhaus [20] in the two dimensional case. It relies on some combination of Gagliardo-Nirenberg type and interpolation inequalities. The novelty here is to use a trace-type Sobolev inequality (simple in the one-dimensional setting) which is required due to the location of the chemical source on the boundary.

We compute the evolution of the density entropy as following,

$$\begin{aligned} \frac{d}{dt} \int_{x>0} n(t, x) \log n(t, x) dx &= \int_{x>0} \partial_t n(t, x) \log n(t, x) dx \\ &= - \int_{x>0} (\partial_x n(t, x) + n(t, 0)n(t, x)) \frac{\partial_x n(t, x)}{n(t, x)} dx \\ &= - \int_{x>0} (\partial_x \log n(t, x))^2 n(t, x) dx + n(t, 0)^2. \end{aligned} \quad (8)$$

The one-dimensional trace inequality we mentioned above writes as following,

$$\begin{aligned} n(t, 0) &= - \int_{x>0} \partial_x n(t, x) dx = - \int_{x>0} (\partial_x \log n(t, x)) n(t, x) dx, \\ n(t, 0)^2 &\leq M \int_{x>0} (\partial_x \log n(t, x))^2 n(t, x) dx. \end{aligned} \quad (9)$$

Therefore we deduce that

$$\frac{d}{dt} \int_{x>0} n(t, x) \log n(t, x) dx \leq (M - 1) \int_{x>0} (\partial_x \log n(t, x))^2 n(t, x) dx, \quad (10)$$

hence the entropy is nonincreasing when the mass is smaller than 1. Observe that equality holds in the trace inequality (9) if $\log n(t, x)$ is constant w.r.t. x : there exists $\alpha(t) > 0$ such that $n(t, x) = M\alpha(t) \exp(-\alpha(t)x)$. In fact the boundary condition (4) implies $M = 1$, which is the only configuration where a stationary state can exist (see Section 2).

It is compulsory to guarantee that the boundary value $n(t, 0)$ makes perfect sense. In fact we simply have:

$$(n(t, x) - n(t, y))^2 \leq \left(\int_x^y n(t, z) dz \right) \left(\int_x^y (\partial_x \log n(t, z))^2 n(t, z) dz \right). \quad (11)$$

The bound (10) guarantees that the dissipation $\int_{x>0} (\partial_x \log n(t, x))^2 n(t, x) dx$ is finite almost every time (provided that some moment remains bounded). Therefore $n(t, \cdot)$ is continuous almost every time.

In fact this estimate can be improved. Combining (10) with suitable bounds on the second moment, we have a local in time $L \log L$ estimate: $\int_{x>0} n(t, x) (\log n(t, x))_+ dx \in L^\infty(0, T)$. This *a priori* knowledge enables to improve (11):

$$(\phi(n(t, x)) - \phi(n(t, y)))^2 \leq \left(\int_x^y n(t, z) (\log n(t, z))_+ dz \right) \left(\int_x^y (\partial_x \log n(t, z))^2 n(t, z) dz \right), \quad (12)$$

where ϕ is a superlinear convex function defined as $\phi'(x) = (\log x)_+^{1/2}$. In particular, $\phi(n(t, 0))^2 \leq C_T \int_{x>0} (\partial_x \log n(t, z))^2 n(t, z) dz$. Hence plugging the last estimate in (8) yields:

$$\frac{d}{dt} \int_{x>0} n(t, x) \log n(t, x) dx \leq -C_T^{-1} \phi(n(t, 0))^2 + n(t, 0)^2.$$

This provides some L^2 estimate for $n(t, 0)$ (locally in time).

To complete the proof of existence it is necessary to gain compactness from the above *a priori* estimates. This usually results from an Aubin-Lions type argument [1, 6]. We refer to [9] for details.

To conclude this Section, let us mention that it is now classical to prove suitable regularizing effects acting on (3) in the subcritical case $M < 1$. Indeed an *a priori* estimate (10) on $\int_{x>0} n(t, x) (\log n(t, x))_+ dx$ yields the boundedness of all L^p -norms ($1 < p < +\infty$) [20, 7, 6, 10].

2 Long-time behaviour: convergence in relative entropy for the critical and the subcritical cases

The critical case. Equilibrium configurations for the density are only possible when the mass is critical: $M = 1$ (as it is for the two-dimensional Keller-Segel problem). In this case, a straightforward computation leads to the one-parameter family:

$$h_\alpha(x) = \alpha e^{-\alpha x}, \quad \alpha > 0. \quad (13)$$

On the other hand, notice that the first momentum of the density is conserved (7). This prescribes a unique choice for α : $\alpha^{-1} = \mathbf{J}(0)$.

Theorem 4 *Assume $n_0(x)$ being as in Theorem 1, and the mass being critical: $M = 1$. As time goes to infinity, the density $n(t, x)$ converges (in relative entropy) towards $h_\alpha(x)$.*

The convergence proof is based on evaluating the time evolution of the relative entropy, defined as follows:

$$\mathbf{H}(t) = \int_{x>0} \frac{n(t, x)}{h_\alpha(x)} \log \left(\frac{n(t, x)}{h_\alpha(x)} \right) h_\alpha(x) dx. \quad (14)$$

The precise description of the equality cases for inequality (9) enables to perform accurate estimates. A direct computation yields the following estimate:

$$\frac{d}{dt} \mathbf{H}(t) = - \int_{x>0} (\partial_x \log n(t, x) + n(t, 0))^2 n(t, x) dx. \quad (15)$$

We refer to [9] for more details.

Self-similar decay in the subcritical case. In the sub-critical case $M < 1$ one expects the density $n(t, x)$ to decay self-similarly. For this purpose the density is appropriately rescaled:

$$n(t, x) = \frac{1}{\sqrt{1+2t}} u \left(\log \sqrt{1+2t}, \frac{x}{\sqrt{1+2t}} \right).$$

The new density $u(\tau, y)$ satisfies:

$$\partial_\tau u(\tau, y) = \partial_{yy} u(\tau, y) + \partial_y (yu(\tau, y)) + u(\tau, 0) \partial_y u(\tau, y), \quad (16)$$

and no-flux boundary conditions: $\partial_y u(\tau, 0) + u(\tau, 0)^2 = 0$. The additional left-sided drift contributes to confine the mass in the new frame (τ, y) . The stationary equilibrium in this new setting can be computed explicitly. The expected self-similar profile writes: $g_\alpha(y) = \alpha \exp(-\alpha y - y^2/2)$, where α is given by the relation $P(\alpha) = M$, P being an increasing function defined as following:

$$P(\alpha) = \int_{y>0} \exp \left(-y - \frac{y^2}{2\alpha^2} \right) dy, \quad \begin{cases} \lim_{\alpha \rightarrow 0} P(\alpha) = 0 \\ \lim_{\alpha \rightarrow +\infty} P(\alpha) = 1 \end{cases}.$$

Theorem 5 Assume $n_0(x)$ being as in Theorem 1, and the mass being subcritical: $M < 1$. As time goes to infinity, the first momentum $\mathbf{J}(\tau)$ of the density converges to $\alpha(1-M)$ and the density $n(t, x)$ converges (in relative entropy) towards $g_\alpha(y)$.

The proof of this Theorem relies again on the time evolution of the relative entropy:

$$\mathbf{H}(\tau) = \int_{y>0} \frac{u(\tau, y)}{g_\alpha(y)} \log \left(\frac{u(\tau, y)}{g_\alpha(y)} \right) g_\alpha(y) dy. \quad (17)$$

More precisely we have:

$$\begin{aligned} & \frac{d}{d\tau} \left\{ \mathbf{H}(\tau) + \frac{1}{2(1-M)} (\mathbf{J}(\tau) - \alpha(1-M))^2 \right\} \\ &= - \int_{y>0} u(\tau, y) (\partial_y \log u(\tau, y) + y + u(\tau, 0))^2 dy - \frac{1}{(1-M)} \left(\frac{d}{d\tau} \mathbf{J}(\tau) \right)^2. \end{aligned} \quad (18)$$

3 Analysis of a coupled ODE/PDE model

We move in this section to a variant of (3) which is more relevant for modelling purposes (see [15] for). In this new setting, the attractive field results from an exchange between *free particles* (having density $n(t, x)$) and *frozen particles* $\mu(t)$ at the boundary $x = 0$:

$$\begin{cases} \partial_t n(t, x) = \partial_{xx} n(t, x) + \mu(t) \partial_x n(t, x), & t > 0, x \in (0, +\infty), \\ \mu'(t) = n(t, 0) - \mu(t), \end{cases} \quad (19)$$

together with the initial conditions: $n(t = 0, x) = n_0(x) \geq 0$ and $\mu(t = 0) = \mu_0$. The conservation of the total mass of particles:

$$\int_{x>0} n(t, x) dx + \mu(t) = M,$$

yields the following boundary condition for the density:

$$\partial_x n(t, 0) + \mu(t) n(t, 0) = \mu'(t). \quad (20)$$

Clearly the concentration of frozen particles $\mu(t)$ remains bounded. Thus attraction cannot be too strong and one expect global existence for (19)–(20). We precise this fact below.

Long-time convergence in the case $M > 1$. We denote by $m(t)$ the mass of the free particles density $n(t, x)$:

$$m(t) = \int_{x>0} n(t, x) dx. \quad (21)$$

(notice $m'(t) + \mu'(t) = 0$ due to the conservation of mass). Introduce the relative entropy:

$$\mathbf{H}(t) = \int_{x>0} \frac{n(t, x)}{m(t)h(x)} \log \left(\frac{n(t, x)}{m(t)h(x)} \right) h(x) dx,$$

where the expected profile h is given by:

$$h(x) = \bar{\mu} \exp(-\bar{\mu}x), \quad \bar{\mu} = M - 1.$$

Theorem 6 As time goes to infinity, the mass $m(t)$ of the density converges to 1 and the density $n(t, x)$ converges (in relative entropy) towards $h(x)$.

The proof of this Theorem relies again on the time evolution of the relative entropy. This is strongly inspired from the previous computation, but takes into consideration the non-conservation of mass for $n(t, x)$ and the dynamics of $\mu(t)$ (we refer to [9] for details).

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