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A Lyapunov function for a two-chemical version of the chemotaxis model *

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In honor of Björn

Abstract.

We answer partially the question of global existence for a chemotaxis model involving two chemical species: a chemo-attractant and a stimulant. We introduce a Lyapunov function for this system and we show that it is non-decreasing assuming a family of threshold conditions.

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Key words: Chemotaxis, bacterial motility, global existence, Lyapunov function.

1 Introduction – Statement of the problem

In the early 70's Keller and Segel have published seminal papers about the modeling of cell populations movements [21, 20] (see also [27]). They describe how chemotaxis may explain several features of cell movement. Chemotaxis means movement in response to a chemical cue, say for instance food, poison or any chemical signal. Various examples of such oriented movement can be found in bacterial motility (*Escherichia Coli* [2]), or in collective cell organization (the slime mold amoebae *Dictyostelium discoideum* and the cAMP molecule [17]). Chemotaxis is also involved, with more specific ingredients, in modelisation of pattern formation [28], vascular network formation [14, 29, 11] and angiogenesis [22, 25]. From a mathematical point of view, the interest of the Keller-Segel system stems from its nonlinear conservative structure which allows for blow-up, critical spaces, traveling waves...

In this paper we will focus on some variant of the classical Patlak, Keller and Segel model (see below for a brief overview of this model). It has been introduced by Brenner *et al.*, and promoted by Tyson *et al.* [6, 30] (see also [24]) in order to explain complex bacterial pattern formation in semi-solid medium [4]. The main additional feature is a second reactant, namely the stimulant f which is consumed by the cells n to produce c. We present here a simplified version

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which captures in a certain sense the main difficulties brought by the additional equation. Namely, we consider the system

$$\left\{\begin{array}{rcl} \dfrac{\partial n}{\partial t} & = & \Delta n - \chi \nabla \cdot (n \nabla c) \quad t \geq 0\,,\ x \in \Omega \subset \mathbb{R}^2, \\ -\Delta c & = & nf - < nf >, \\ \dfrac{\partial f}{\partial t} & = & -nf. \end{array}\right. \tag{1.1}$$

We complete this system with initial conditions n^0, c^0, f^0. Here Ω is a bounded domain, and we consider zero-flux boundary conditions for both n and c.

The Patlak, Keller and Segel (PKS) model. The PKS system consists of two coupled equations for the evolution of the cell density $n(t,x)$ and the chemoattractant $c(t,x)$ respectively. Cell density is governed by a drift-diffusion equation

$$\frac{\partial n}{\partial t} + \nabla \cdot \Big(- \nabla n + \chi n \nabla c \Big) = 0\,, \quad t \geq 0\,,\ x \in \Omega \subset \mathbb{R}^2, \tag{1.2}$$

and the concentration of chemical satisfies a reaction-diffusion equation wich reads

$$-\Delta c = n - < n >, \tag{1.3}$$

in the limiting case of fast diffusion [18]. Here $< n >$ denotes the mean value of n over the domain Ω. Boundary conditions are zero-flux. The key parameters are χ the chemotactic sensitivity – which is assumed to be constant here – and M the total mass of cells – which is formally conserved. The general behaviour of this system is now quite well understood, and main results are summarized in the following theorem [13, 5, 15, 10].

THEOREM 1.1 (GLOBAL EXISTENCE FOR THE PKS MODEL). *Assume Ω is a regular bounded domain and $n_0 \in L^\infty(\Omega)$. If $\chi M < 4\pi$ solutions are global in time. If $\chi M > 8\pi$ and the second moment of n_0 is large enough then the solution blows-up in finite time.*

If Ω is the whole space $\mathbb{R}^2$ and both $n_0\big(|\log n_0| + (1+|x|^2)\big) \in L^1$, then solutions are global in time if $\chi M < 8\pi$, or blow up in finite time if $\chi M > 8\pi$.

The existence parts of these results are based on two different strategies. The common feature is to prove equi-integrability for the cell density n thanks to *a priori* estimates. These *a priori* estimates are of two types. Given a functional Φ growing faster than linearly (typically $\Phi(u) = u \log u$ or $\Phi(u) = u^p$), a direct computation of $\frac{d}{dt} \int \Phi(n)\, dx$ gives two terms of opposite signs. In general a Gagliardo-Nirenberg-Sobolev (GNS) inequality can be used to estimate the balance between the diffusion and the chemotactic contributions. The threshold condition coming from GNS inequality has the right homogeneity, but it is not optimal. On the other hand, the free energy for system (1.2), (1.3) writes

$$\mathcal{E}(t) = \int n \log n - \frac{\chi}{2} \int cn.$$

It is non increasing and therefore it is possible to estimate $\int n \log n$ if the two opposite contributions can be compared to each other, by means of fine inequalities (namely Trudinger-Moser or logarithmic Hardy-Littlewood-Sobolev inequalities).

There are several shortcomings to model (1.2), (1.3). Several biochemistry aspects are not taken into account, as well the type of physical support for the experiments [24, Vol. II chap. 5]. Also concentrations points do not move in usual numerical simulations [23] by opposition to experimental observations or numerical simulations of (1.1) as shown in Figure 1.1. The formulas for the aggregate motion in [31] confirm that different regularizations of (1.2), (1.3) give different dynamics.

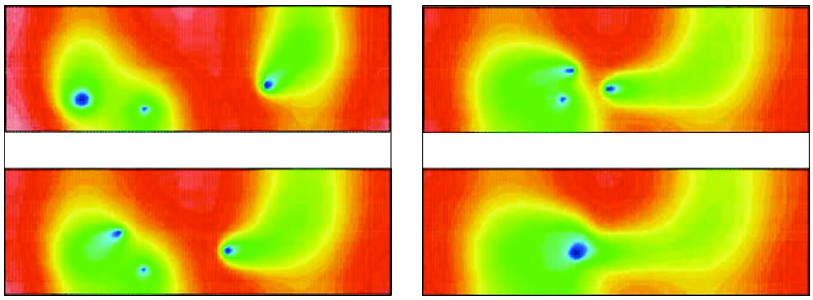

Figure 1.1: Motion of three aggregates in an extended Keller-Segel system as (1.1). Courtesy of A. Marrocco (work in preparation).

Statement of the main result. As we mentioned above, a natural question arising among various studies of PKS models is whether solution blows up or not in finite time. Various modifications of the classical model (1.2), (1.3) have already been proposed to prevent formation of singularities. For instance, Kowalczyk analyzed a system including nonlinear cell diffusion [19]. On the other side Painter and Hillen considered a saturating effect on the chemotactic sensitivity [16]: $\chi(n)$ vanishes for large n (see also [7] for a discussion concerning these volume effects). In the following we raise the question of any blow-up for the model (1.1), a question that we only partially answer here. Notice that (1.1) can be considered as an extension of some system arising to describe angiogenesis that has been studied in [9], and for which the question of blow up is also open.

First let us present a method, directly inspired from [18]. It is based on the following Gagliardo-Nirenberg-Sobolev inequality [12, 26],

$$\int n^2 \le \mathcal{C}_{gns} \int n \int |\nabla \sqrt{n}|^2. \tag{1.4}$$

This estimation appears naturally when computing the evolution of $\int n \log n$.

$$\begin{aligned}\frac{d}{dt}\int n\log n &\leq -4\int |\nabla\sqrt{n}|^2 + \chi\int n^2 f \\ &\leq -4\int |\nabla\sqrt{n}|^2 + \chi\|f\|_\infty \int n^2 \\ &\leq \big(-4+\chi\|f\|_\infty M\mathcal{C}_{gns}\big)\int |\nabla\sqrt{n}|^2.\end{aligned}$$

Therefore, using this only ingredient, we can hope to conclude only under the condition

$$\chi\|f^0\|_\infty M < \mathcal{C}^*. \tag{1.5}$$

This condition is not satisfactory in the sense that it doesn't bring anything new by comparison to the classical Keller and Segel model, and because it doesn't capture the fine coupling with the additional equation $\partial_t f = -nf$.

Unfortunately we know no energy structure, wich makes this model dramatically different from (1.2), (1.3). However we derive in this paper a new class of conditions involving the parameters χM and $\|f^0\|_\infty$ which guarantees *a priori* estimates.

THEOREM 1.2 (A PRIORI ESTIMATES FOR THE EXTENDED PKS MODEL). *Let Ω be a bounded domain. It exists a family of conditions*

$$\chi\|f^0\|_\infty M^{1-\lambda} \leq C_\lambda, \tag{1.6}$$

indexed by $\lambda \in [0, \frac{1}{4}]$ such that: if at least one of these conditions is fulfilled, then the solution of (1.1) *is* a priori *globally equi-integrable.*

From now on our strategy consists in studying the variations of a well-chosen functional $\mathcal{W}$ combining with homogeneity the standard energy of PKS equations and that of the angiogenesis model used in [9], namely

$$\mathcal{W}(t) = \int n\log n + b\int nf^\gamma + \frac{a}{2}\int |\nabla f^\delta|^2,$$

where a, b are some constants depending on the parameters, and δ, γ are exponents without homogeneity.

At the end of this contribution we are concerned with the problem of global existence for the system (1.1), and we provide some discussion how to prove it based on theorem 1.2.

THEOREM 1.3 (GLOBAL EXISTENCE FOR THE EXTENDED MODEL). *Assume condition 1.6, $n^0 \in L^1 \cap L^p$ for some $p > 1$, and $\mathcal{W}(0)$ is finite, then there is a unique weak solution to* (1.1) *that satisfies $\mathcal{W}(t) \leq \mathcal{W}(0)$ and $n \in L^\infty\big(\mathbb{R}_+; L^1 \cap L^p\big)$.*

In section 2 we present some regularization estimates which justify the choice of the functional $\mathcal{W}$. In sections 3 and 4 we drive the calculation leading to $\frac{d}{dt}\mathcal{W} \leq 0$, and hence we prove theorem 1.2. Finally we prove theorem 1.3 in section 5.

2 Preliminaries: global existence for the chemotaxis and the angiogenesis models

In this section we review some basics of the existence theorems for the chemotaxis and the angiogenesis models respectively. The energy structure for the first one has already been described in introduction.

2.1 The chemotaxis model

We first highlight the following statement

$$\frac{d}{dt}\int n^p + 2\frac{p-1}{p}\int |\nabla n^{p/2}|^2 \leq \frac{\chi^2 p(p-1)}{2}\|\nabla c\|^2_{L^\infty_{t,x}}\int n^p, \tag{2.1}$$

which comes directly from (1.2). It means that some estimate on ∇c cancels the nonlinearity and provide any L^p bound, $p < \infty$, for the cell density n. Unfortunately this ∇c estimate is not available *e.g.*, and the usual way is to start from equi-integrability, namely $\int n|\log n|\,dx \leq C$, which avoids the formation of Dirac masses; then to propagate L^p bounds thanks to the following computation

$$\begin{aligned}\frac{d}{dt}\int (n-k)_+^p + 4\frac{p-1}{p}\int |\nabla (n-k)_+^{p/2}|^2 \\ \leq \chi(p-1)\int (n-k)_+^{p+1} + O\Big(\int (n-k)_+^p\Big).\end{aligned} \tag{2.2}$$

Indeed using the Gagliardo-Nirenberg-Sobolev inequality

$$\int (n-k)_+^{p+1}\,dx \leq \mathcal{C}_{gns}(p)\int |\nabla (n-k)_+^{p/2}|^2\,dx\int (n-k)_+\,dx,$$

and equi-integrability yelds the inequality

$$\frac{d}{dt}\int (n-k)_+^p \leq O\Big(\int (n-k)_+^p\Big),$$

which ensures that $\|n\|_{L^p}$ is controlled [18, 9]. This argumentation remains valid in the extended system (1.1), so we are to prove equi-integrability only.

2.2 The angiogenesis model

Following [1] a simplified model for angiogenesis has been proposed in [8]. It reads

$$\left\{\begin{array}{rcl}\dfrac{\partial n}{\partial t} &=& \Delta n - \chi\nabla\cdot(n\nabla f) \quad t\geq 0\,,\ x\in\Omega\subset\mathbb{R}^2,\\ \dfrac{\partial f}{\partial t} &=& -nf,\end{array}\right. \tag{2.3}$$

where n denotes the endothelial cell density, and f denotes some chemical angiogenic factor, secreted by a tumor for instance. Boundary conditions are zero-flux

as well. Contrary to the chemotactic model PKS, this system admits a positive energy structure $\frac{d}{dt}\mathcal{E} \leq 0$ with

$$\mathcal{E}(t) = \int n \log n + 2\chi \int |\nabla \sqrt{f}|^2.$$

This energy structure provides us with global existence of weak solutions. It is also possible to derive L^p bounds for the cell density similarly to (2.2) under smallness assumptions, but we won't enter into details here (see [9]).

3 Outline of the calculation

We consider a combination of the following type [13]

$$\mathcal{W}(t) = \int n \log n + b \int n f^\gamma + \frac{a}{2} \int |\nabla f^{\gamma-1}|^2. \tag{3.1}$$

Our goal is to show that it is decreasing for suitable values of a, b, which are proved to exist whenever $(\chi \|f^0\|_\infty)^\gamma M^{\gamma-1} \leq C_\gamma$, $\gamma \geq 4$. Note that we will keep γ along the paper, and we will introduce $\lambda = \gamma^{-1}$ in conclusion. We first compute each term of the derivative $\dfrac{d}{dt}\mathcal{W}$.

$$\frac{d}{dt} \int n \log n \leq -4 \int |\nabla \sqrt{n}|^2 + \chi \int n^2 f, \tag{3.2}$$

$$b \frac{d}{dt} \int n f^\gamma = -b \int \nabla n \cdot \nabla f^\gamma + \chi b \int n \nabla c \cdot \nabla f^\gamma - \gamma b \int n^2 f^\gamma, \tag{3.3}$$

$$\frac{a}{2} \frac{d}{dt} \int |\nabla f^{\gamma-1}|^2 = -\frac{\gamma-1}{2} a \int \nabla n \cdot \nabla f^{2\gamma-2} - (\gamma-1) a \int n |\nabla f^{\gamma-1}|^2. \tag{3.4}$$

In order to compensate the bad influence of the positive (*resp.* no-sign) terms in the right-hand side of (3.2) (*resp.* (3.3), (3.4)), we plan to associate them with the negative ones in two ways. The first group is made of

$$-4 \int |\nabla \sqrt{n}|^2 - \left\{ \begin{array}{c} b \displaystyle\int \nabla n \cdot \nabla f^\gamma \\ \dfrac{\gamma-1}{2} a \displaystyle\int \nabla n \cdot \nabla f^{2\gamma-2} \end{array} \right\} - (\gamma-1) a \int n |\nabla f^{\gamma-1}|^2 \leq 0. \tag{3.5}$$

The sign of this expression will be determined thanks to a recombination into a remarkable square. The second group is made of

$$\chi \int n^2 f - \gamma b \int n^2 f^\gamma.$$

The non-friendly term $\chi b \int n \nabla c \cdot \nabla f^\gamma$ plays an ambivalent role in this description, because it gives contributions to each group.

4 Details of the estimation

Our main objective is to preserve the homogeneity along computations. For this purpose we frequently introduce some homogeneity constant which has to be fixed later on.

4.1 The first group (3.5)

We force a remarkable square to appear thanks to the square terms. We are able to upperbound (3.5) by

$$-4\int|\nabla\sqrt{n}|^2+2b\frac{\gamma}{\gamma-1}\|f\|_\infty\int|\nabla\sqrt{n}|\cdot|\sqrt{n}\nabla f^{\gamma-1}|-(\gamma-1)a\int n|\nabla f^{\gamma-1}|^2\leq 0.$$

So a first condition concerning the homogeneity of a and b comes naturally for this expression to be non-positive: the discriminant is non-positive;

$$\left(2b\|f(t)\|_\infty\frac{\gamma}{\gamma-1}\right)^2-16a(\gamma-1)\leq 0. \tag{4.1}$$

More precisely we choose the constants a, b so that

$$\left(2b\|f^0\|_\infty\frac{\gamma}{\gamma-1}\right)^2=a(\gamma-1),\quad \gamma>1. \tag{4.2}$$

The same computation arises for the other expression of (3.5), namely

$$-4\int|\nabla\sqrt{n}|^2+2(\gamma-1)a\int|f|^{\gamma-1}|\nabla\sqrt{n}||\sqrt{n}\nabla f^{\gamma-1}|-(\gamma-1)a\int n|\nabla f^{\gamma-1}|^2\leq 0.$$

We get a similar condition on the discriminant of this expression, and we choose exactly the constant a to be

$$4(\gamma-1)a\|f^0\|_\infty^{2\gamma-2}=1, \tag{4.3}$$

and the combination of (4.2) and (4.3) gives in addition

$$4\left(b\|f^0\|_\infty^\gamma\frac{\gamma}{\gamma-1}\right)=1. \tag{4.4}$$

In this subsection we have hidden the two terms $b\nabla n\cdot\nabla f^\gamma$ and $\frac{\gamma-1}{2}a\int\nabla n\cdot\nabla f^{2\gamma-2}$ in the negative contributions of

$$-2\int|\nabla\sqrt{n}|^2\quad\text{and}\quad-\frac{1}{2}(\gamma-1)a\int n|\nabla f^{\gamma-1}|^2.$$

4.2 Estimating the ambivalent term $\int n\nabla c\cdot\nabla f^\gamma$

We can combine this no-sign term in a general way

$$\begin{aligned}\int n|\nabla c\cdot\nabla f^\gamma| &= \frac{\gamma}{\gamma-1}\int nf|\nabla c\cdot\nabla f^{\gamma-1}|,\\ &\le \left(\frac{\gamma}{\gamma-1}\right)^2\frac{K}{2}\int n|\nabla f^{\gamma-1}|^2+\frac{1}{2K}\int nf^2|\nabla c|^2,\end{aligned}\tag{4.5}$$

with a homogeneity constant K which has to be fixed. We would like to associate the first right-hand side term with $-(\gamma-1)a\int n|\nabla f^{\gamma-1}|^2$, that is

$$\chi\frac{K}{2}b\left(\frac{\gamma}{\gamma-1}\right)^2\int n|\nabla f^{\gamma-1}|^2-\frac{1}{2}a(\gamma-1)\int n|\nabla f^{\gamma-1}|^2\le 0.$$

In fact we choose

$$\chi b\left(\frac{\gamma}{\gamma-1}\right)^2K=(\gamma-1)a,\quad\text{hence}\quad 4\chi b\left(\frac{\gamma}{\gamma-1}\right)^2\|f^0\|_\infty^{2(\gamma-1)}K=1,\tag{4.6}$$

(after combination with (4.3)). The second right-hand side term of (4.5) will be eliminated thanks to the combination of a Sobolev inequality

$$\|\nabla c\|_4^4\le\mathcal{C}_{\mathcal{S}}\|nf\|_{4/3}^4,\tag{4.7}$$

and a Gagliardo-Nirenberg-Sobolev inequality

$$\left(\int n^{4/3}\right)^3\le\mathcal{C}_{gns}M^3\int|\nabla\sqrt{n}|^2.$$

Notice that this constant $\mathcal{C}_{gns}$ differs from (1.4): we just want to mention the origin of the constant in the following. It comes

$$\int nf^2|\nabla c|^2\le\frac{L}{2}\int n^2f^4+\frac{1}{2L}\int|\nabla c|^4,\tag{4.8}$$

and also

$$\int|\nabla c|^4\le\mathcal{C}^*\|f(t)\|_\infty^4M^3\int|\nabla\sqrt{n}|^2,\tag{4.9}$$

where $\mathcal{C}^*=\mathcal{C}_{\mathcal{S}}\mathcal{C}_{gns}$. To compare (4.9) with our available negative term $-4\int|\nabla\sqrt{n}|^2$ from (3.2), parameters should fulfill

$$\chi bK^{-1}L^{-1}\|f\|_\infty^4M^3\mathcal{C}^*\le 16.$$

We choose precisely

$$\chi bK^{-1}L^{-1}\|f^0\|_\infty^4M^3\mathcal{C}^*=4.\tag{4.10}$$

In this subsection we have consumed

$$-\int |\nabla\sqrt{n}|^2 \quad \text{and} \quad -\frac{1}{2}(\gamma-1)a\int n|\nabla f^{\gamma-1}|^2.$$

Finally we have to deal with the last remaining positive terms, namely $\int n^2 f$ in (3.2) and $\int n^2 f^4$ in (4.8).

At this stage we leave chosen the constants: a by (4.3), b by (4.4), K by (4.6), and L by (4.10).

4.3 The second group

In order to eliminate the two terms $\int n^2 f$ and $\int n^2 f^4$, we of course associate them with $\int n^2 f^\gamma$. That is why we impose $\gamma \geq 4$ in theorem 1.2. We use the following majorations which distinguish between high and low values of f.

$$Y \leq R^{-1}\mathcal{C}(\gamma) + R^{\gamma-1}Y^\gamma,$$

$$X^4 \leq S^{-4}\mathcal{C}(\nu) + S^{\gamma-4}X^\gamma\,, \quad 4\nu = \gamma,$$

with the constant $E(\nu) = \mathcal{C}(\nu)^{\nu-1} = \frac{(\nu-1)^{(\nu-1)}}{\nu^\nu}$. Then for each term $\int n^2 f^4$ and $\int n^2 f$ we get two new terms involving $\int n^2$ and $\int n^2 f^\gamma$:

$$\chi\int n^2 f \leq \chi\mathcal{C}(\gamma)R^{-1}\int n^2 + \chi R^{\gamma-1}\int n^2 f^\gamma,$$

$$\chi b\frac{K^{-1}L}{4}\int n^2 f^4 \leq \chi b\frac{K^{-1}L}{4}S^{-4}\mathcal{C}(\nu)\int n^2 + \chi b\frac{K^{-1}L}{4}S^{\gamma-4}\int n^2 f^\gamma.$$

At this stage we can use the first Gagliardo-Nirenberg-Sobolev inequality (1.4) to estimate $\int n^2$, and we deduce the following conditions

$$2\chi\mathcal{C}(\gamma)R^{-1}M\mathcal{C}_{gns} = 1, \quad 2\chi b\frac{K^{-1}L}{4}S^{-4}\mathcal{C}(\nu)M\mathcal{C}_{gns} = 1. \tag{4.11}$$

On the other hand we look for a cancellation of the last positive remaining terms involving both $\int n^2 f$, and therefore we impose

$$2\chi R^{\gamma-1} \leq \gamma b, \quad 2\chi b\frac{K^{-1}L}{4}S^{\gamma-4} \leq \gamma b. \tag{4.12}$$

Finally we have determined all the homogeneity constants introduced in the calculations, and we can restate (4.11), (4.12) as following:

$$2\chi\left(2\chi\mathcal{C}(\gamma)\mathcal{C}_{gns}M\right)^{\gamma-1} \leq \gamma b, \tag{4.13}$$

$$\chi bK^{-1}L\left(\chi bK^{-1}L\mathcal{C}(\nu)\mathcal{C}_{gns}M\right)^{\nu-1} \leq 2^\nu\gamma b, \quad \nu = \frac{\gamma}{4}, \tag{4.14}$$

and we recall that b, K, L are already fixed. Notice that (4.13) and (4.14) are redundent thanks to the exponents involved.

4.4 Consequences of the homogeneity relations

Replacing b, K and L by their values (4.4), (4.6), (4.10), and because we have set $\gamma = 4\nu$, we find that the two redundent conditions (4.13) and (4.14) can summarize simply into the single inequality

$$E(\gamma)\chi^\gamma \|f\|_\infty^\gamma M^{\gamma-1} \leq C_\gamma^\gamma, \tag{4.15}$$

where the constants C_γ are uniformly bounded. Consequently, taking the γ-root of (4.15) and because

$$E(\gamma)^{1/\gamma} = \frac{(1-1/\gamma)^{1-1/\gamma}}{(1/\gamma)^{1/\gamma}}$$

is bounded for $\gamma \geq 4$, we obtain the final condition, announced in theorem 1.2.

$$\chi \|f^0\|_\infty M^{1-\lambda} \leq C_\lambda, \quad \lambda \in [0, \frac{1}{4}],$$

with a bounded family of constants (C_λ). Note that the special case $\lambda = 0$ corresponds to estimation (1.5).

5 Global existence for the system (1.1)

We prove theorem 1.3 and we proceed as usually in three steps.

Step 1. Regularization of the system. We propose to replace the second equation with

$$-\Delta c = T_K(nf) - < T_K(nf) >, \tag{5.1}$$

where $T_K(u) = \min(u, K)$. The corresponding system together with regularized initial conditions is solved using Banach fixed point theorem. The truncature (5.1) ensures that the solution (n_K, c_K, f_K) is global in time, because it avoids formation of any singularity; see section 2.1, and notice that $T_K(nf)$ is a priori bounded in $L^1 \cap L^\infty$, therefore $\nabla c \in W^{1,\infty}$ by Young's inequality.

Step 2. Estimates for the regularized system. *A priori* estimates which have been proved formally in section 4 can be adapted to the regularized system with minor modifications. First we compute the time derivative of $\mathcal{W}$ related to the regularized model.

$$\frac{d}{dt}\int n \log n \leq -4\int |\nabla\sqrt{n}|^2 + \chi \int n T_K(nf), \tag{5.2}$$

$$b\frac{d}{dt}\int nf^\gamma = -b\int \nabla n \cdot \nabla f^\gamma + \chi b \int n \nabla c \cdot \nabla f^\gamma - \gamma b \int n^2 f^\gamma, \tag{5.3}$$

$$\frac{a}{2}\frac{d}{dt}\int |\nabla f^{\gamma-1}|^2 = -\frac{\gamma-1}{2}a\int \nabla n \cdot \nabla f^{2\gamma-2} - (\gamma-1)a\int n|\nabla f^{\gamma-1}|^2. \tag{5.4}$$

Only the first part (5.2) is affected by the truncature. Consequently we are able to follow the consecutive steps of section 4, because the negative terms in (5.2), (5.3) and (5.4) necessary for cancellations remain unchanged.

Step 3. Propagation of regularity. Finally we can use the upperbound of $\int n \log n$ to prove L^p bounds for the cell density (see [18, 9] and section 2). These estimations provide also time compactness in terms of

$$\int_0^\infty \|\nabla n^{p/2}\|^2_{L^2(\Omega)} \leq C(\|n_0\|_{L^p(\Omega)}).$$

Passing to the limit, the main difficulty lies in the nonlinear term $\nabla \cdot (n\nabla c)$ and we need some compactness. It is provided by the Lions–Aubin lemma [3] which claims that the embedding

$$\Big\{u \in L^2\big(0,T;H^1\big),\ \partial_t u \in L^2\big(0,T;L^2\big)\Big\} \hookrightarrow \mathcal{C}\big(0,T;L^2\big)$$

is compact.

6 Conclusion

We have studied a priori bounds (and existence) for the variant (1.1) of the well-known Keller and Segel model. Because of the lack of energy structure we have introduced a new type of functional $\mathcal{W}$ which is decreasing under some condition involving the parameters. These new considerations may be extended to new kinds of models where several extracellular products are involved (angiogenesis for instance).

We have been able to find a new threshold condition (1.6) ensuring that $\int n \log n$ remains bounded and thus that n is equi-integrable. But we have no certitude whether solutions may blow up or not above these thresholds. In fact we have not shown any existence of a blowing-up solution for this system, and the mechanism for such a blow-up is certainly more complex than for the Keller and Segel model.

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