Non-linear analysis of a model for yeast cell communication

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January 7, 2019

Abstract

We study the non-linear stability of a coupled system of two non-linear transport-diffusion equations set in two opposite half-lines. This system describes some aspects of yeast pairwise cellular communication, through the concentration of some protein in the cell bulk and at the cell boundary. We show that it is of bistable type, provided that the intensity of active molecular transport is large enough. We prove the non-linear stability of the most concentrated steady state, for large initial data, by entropy and comparison techniques. For small initial data we prove the self-similar decay of the molecular concentration towards zero. Informally speaking, the rise of a dialog between yeast cells requires enough active molecular transport in this model. Besides, if the cells do not invest enough in the communication with their partner, they do not respond to each other; but a sufficient initial input from each cell in the dialog leads to the establishment of a stable activated state in both cells.

AMS Subject Classification (2010): 35B32, 35B35, 35B40, 35K40, 35Q92, 92B05, 92C17, 92C37

1 Introduction, model and results

How do cells communicate with each other? This question, which seems simple, is still under current investigation. Cell communication plays fundamental role in many cellular processes including cell division and differentiation, directional movement as well as morphogenesis. Defects in cell-cell communication are also implied in the development of cancer.

From the biological point of view, a prototypical model for cell communication is given by yeast cell mating. In this work we propose and analyze a minimal model describing some aspects of yeast cell-cell communication. This model is based on a system of coupled non-linear and transport-diffusion equations. The uncoupled transport-diffusion equation, was introduced and studied in previous works to describe internal protein dynamics in a single yeast cell [21, 9, 8, 28, 32], including the process of spontaneous polarization. Here, from the mathematical viewpoint, the novelty is the coupling between two such equations set on two opposite half-lines. The coupling occurs from some scalar quantities which account for molecular communication between cells.

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In the context of yeast cell communication, the main interest of this model is to link the output of cell communication to protein aggregation on both cell membranes. Indeed, the capacity of a cell to interact with other cells depends on its intracellular protein distribution n_i: proteins located on the cell membrane (μ_i) are able to combine with some exterior signalling pheromone to enhance active transport of proteins to produce some signal that will diffuse in the exterior medium. Hence, in a cell where the proteins are located in its bulk, the communication with the environment will be very difficult. On the contrary, a cell with many proteins on its membrane will have a great capacity for communication.

In the following, each cell $i = 1, 2$ is modelled as a half-line, with a boundary reduced to a single point (see Figure 1). As the coupling between the two cells occurs through a scalar quantity, there is no need to consider disjoint half-lines. So we consider that each cell coincides with $\mathbb{R}_+$, without loss of generality. Each cell is characterized by the distribution of some protein involved in the polarization process. The proteins can be in two states: either in the cell bulk – with density $n_i(t, x)$ – or attached to the cell boundary – with fraction $\mu_i(t)$. Accordingly, the total molecular content in each cell is:

$$M_i := \mu_i(t) + \int_0^\infty n_i(t, x) \, \mathrm{d}x \,. \tag{1.1}$$

The model consists of the following equations:

$$\partial_t n_i(t, x) = \partial_{xx} n_i(t, x) + \chi \, \mu_1(t) \, \mu_2(t) \, \partial_x n_i(t, x) \,, \quad t \geq 0 \,, \quad x \geq 0 \,. \tag{1.2}$$

for $i = 1, 2$, where $n_i = n_i(t, x)$ is the density of some molecular content in each cell bulk, $\mu_i(t)$ ($i = 1, 2$) is the fraction of molecules attached to each cell boundary at $x = 0$, and $\chi > 0$ is a parameter, which accounts for the strength of the coupling between the two cells. This parameter combines the effects of intracellular protein transport, as well as the level of interaction between the two cells that may depend, *e.g.* upon the mutual distance. The dynamics of $\mu_i(t)$ are given by attachment and detachment kinetic at the boundaries:

$$\frac{\mathrm{d}\mu_i}{\mathrm{d}t}(t) = n_i(t, 0) - \mu_i(t) \,, \quad t > 0 \,. \tag{1.3}$$

Moreover, equations (1.2) are complemented with the following flux boundary conditions:

$$\frac{\mathrm{d}\mu_i}{\mathrm{d}t}(t) = \partial_x n_i(t, 0) + \chi \, \mu_1(t) \, \mu_2(t) \, n_i(t, 0) \,, \quad t > 0 \,, \tag{1.4}$$

which ensure the conservation of the total molecular content in each cell (1.1),

$$M_i(t) = M_i(0) = \mu_i(0) + \int_0^\infty n_i^0(x) dx = M_i.$$

Briefly, the main assumptions that lead to (1.2) are the following: (i) Proteins in the bulk can be either freely diffusing or actively transported towards the cell boundary, (ii) The effects of activated proteins attached to the boundary (μ_i) are twofolds as they enhance active transportation in both cells. The underlying processes are: nucleation of actin filaments in one cell, and secretion of some external signalling molecule (pheromone) which can bind to the other cell. We refer to Section 2 for a detailed presentation of the model with biological motivations.

For the sake of simplicity, throughout this work we will assume that both cells have the same (conserved) total molecular content which we normalize as follows:
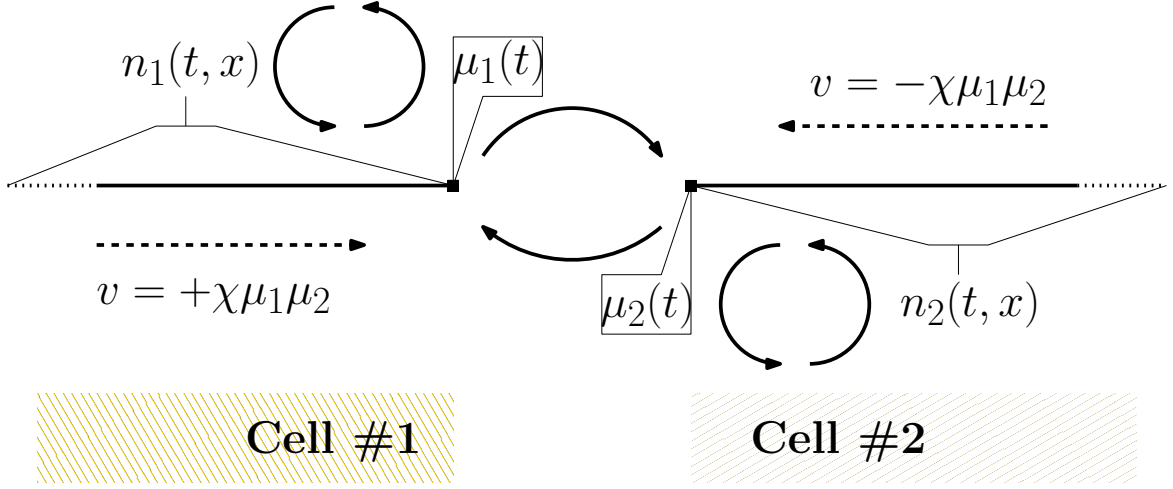
$$M_1 = M_2 = 1 \,. \tag{1.5}$$

Figure 1: Cartoon of the model (1.2)-(1.3)-(1.4): the molecular content in the bulk experiences diffusion and transport at speed $v = \pm\chi\mu_1\mu_2$, where μ_i can be seen as the fraction of activated molecules attached to the cell boundary. The dialog between the two cells arises as they secrete some communication signal (a pheromone) at a rate proportional to μ_i, enhancing further the transport of molecular content in the opposite cell. Details are given in Section 2.

Since the transport speed is bounded, $\mu_i(t) \leq M_i$, global existence of solutions to the Cauchy problem (1.2)–(1.3)–(1.4) holds true. We refer to Appendix C for the sketch of the well-posedness theory for the PDE-ODE coupled system. Here, our aim is to precise the long time behaviour. In order to bypass the lack of comparison principle on equation (1.2), our method is based on a concentration-comparison principle that is obtained when equation (1.2) is integrated in space, see [28]. This principle allows constructing some remarkable sub/supersolutions and performing a non-linear stability analysis.

As the coupling goes through a cubic nonlinearity in the transport term of (1.2), it is expected that for small initial data, or alternatively for small parameter χ, the system behaves essentially linearly. Indeed, we identified the threshold $\chi \lessgtr 4$ above which non-linear effects are at play.

Before stating the results, we start with some notations. For $i = 1, 2$, the function N_i is defined as the cumulated distribution function of $\mu_i(t)\delta_0 + n_i(t, x)\,\mathrm{d}x$:

$$N_i(t, x) = \mu_i(t) + \int_0^x n_i(t, y)\,\mathrm{d}y\,. \tag{1.6}$$

Besides, for any parameter $\mu \in (0, 1)$ the cumulated distribution function N_μ is defined by

$$N_\mu(x) = \mu + (1 - \mu)\left(1 - e^{-\chi\mu^2 x}\right)\,.$$

Suppose that $\chi \geq 4$. Then the polynomial P defined by

$$P(X) = \chi X^2 - \chi X + 1\,, \tag{1.7}$$

admits two real roots denoted by $\mu_- \leq \mu_+$. The functions $n_\pm$ and $m_\pm$ are defined by:

$$n_\pm(x) = (1 - \mu_\pm)m_\pm(x) = (1 - \mu_\pm)\chi\mu_\pm^2 e^{-\chi\mu_\pm^2 x}\,, \tag{1.8}$$

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Before stating our results about the analysis of (1.2)–(1.3)–(1.4), let us briefly comment the literature. The uncoupled problem, corresponding to a single cell ($i = 1$, say) in a constant field of pheromone concentration ($\mu_2 \equiv 1$, say) was introduced in [9] and [8] as the one-dimensional version of a class of active transport models introduced previously in [21] for the modelling of cell polarization. Two cases were considered: either a direct coupling $\mu_i(t) = n_i(t,0)$, or a coupling via exchange of molecular content at the boundary, as (1.3). In the former case, the solution may blow up in finite time for large χ, due to the strong interaction. In the latter case, the solution converges to a bounded stationary state for large χ, but decays to zero for small χ. The results were improved later in [28] using suitable comparison principles.

This dichotomy between concentration of the solution vs. self-similar decay is analogous to the classical Keller-Segel equation for chemotaxis [5]. However, the interaction here goes through the trace value at the boundary $n_i(t,0)$, which makes the analysis more difficult. It is more singular, and furthermore it lacks symmetry properties and in particular there is no free energy associated with (1.2)–(1.3)–(1.4) to our knowledge. Nevertheless, we show here that the system inherits some structure from this analogy. In particular, it is possible to construct Lyapunov functionals that contain some entropy contributions. This is quite remarkable as the system is genuinely non-linear.

Finally, let us mention that similar models involving a coupling between a one-dimensional PDE and a scalar boundary value appear in the modelling of NNLIF models [6, 10, 11, 16, 7], except that the derivative at the boundary is involved, among other differences. Nevertheless, entropy techniques have also been used in this context, as well as moment estimates.

All the aforementioned works deal with only one density (of proteins or attracting cells or firing neurons). In the present work we show that many results can be extended to the nonlinear coupling between two protein densities, as in Figure 1. Besides, the underlying cubic nonlinearity yields a bistable behaviour which deserves some careful analysis.

The following preliminary statement is concerned with the linear stability analysis of the system.

Proposition 1.1 (Steady states and linear stability)**.** *Consider the system* (1.2)–(1.3)–(1.4) *and* (1.5). *Any steady state* $(\overline{n}_i, \overline{\mu}_i)$ *satisfies* $\overline{n}_1 = \overline{n}_2$ *and* $\overline{\mu}_1 = \overline{\mu}_2$. *Moreover, the following alternative holds true:*

- *If $\chi < 4$, then the system does not admit a non-zero steady state.*

- *If $\chi > 4$, then, there are two steady states (n_-, μ_-) and (n_+, μ_+). Furthermore, (n_+, μ_+) is linearly stable while (n_-, μ_-) is linearly unstable.*

- *If $\chi = 4$, then $\mu_- = \mu_+ = 1/2$ and the sytem admits a unique steady state.*

Intuitively, the system is of bistable type, with a bifurcation at $\chi = 4$. For $\chi < 4$, the density is expected to converge to zero for any initial condition and to follow the self similar behaviour driven by the diffusion part. For $\chi > 4$ it is expected either to follow the latter self similar behaviour or to converge to the most concentrated steady state (n_+, μ_+) defined by (1.7), (1.8), depending on the initial data (see Table 1).

In this work we are interested in making this informal statement rigorous in the non-linear regime. We shall refer to our previous work [8] for well-posedness of the system in the uncoupled case ($\mu_2 \equiv 1$), including the careful handling of the trace term $n(t,0)$.

Our main result is the convergence towards the steady state (n_+, μ_+) defined by (1.7), (1.8), for large enough initial data.

Proposition 1.2 (Non-linear stability of the largest equilibrium)**.** *Assume $\chi > 4$ and let $(n_\pm, \mu_\pm)$ be defined by* (1.7), (1.8). *Let $(n_i, \mu_i)_{1 \le i \le 2}$ be the solution to* (1.2)–(1.3)–(1.4) *with initial data*

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$(n_i^0, \mu_i^0)_{1 \le i \le 2}$ satisfying (1.5). Suppose that both the entropy and the first moment of the initial data $(n_i^0, \mu_i^0)_{1 \le i \le 2}$ are finite: $\int_0^\infty n_i^0(x)(x + \log n_i^0(x))\,\mathrm{d}x < +\infty$. Assume in addition that there exist two real numbers $(\underline{\mu}_0, \overline{\mu}_0) \in (0, 1)^2$ ordered as follows,

$$\mu_- < \underline{\mu}_0 < \mu_+ < \overline{\mu}_0\,, \tag{1.9}$$

such that the initial data are ordered as follows:

$$(\forall x \ge 0) \quad N_{\underline{\mu}_0}(x) \le N_i^0(x) \le N_{\overline{\mu}_0}(x)\,, \quad i = 1, 2\,. \tag{1.10}$$

Then, for $i = 1, 2$, the convergence of (n_i, μ_i) towards the steady state (n_+, μ_+) holds true in the following sense:

$$\begin{cases} |\mu_i - \mu_+| = O\left(\frac{1}{\sqrt{1+t}}\right)\,, \\ \|n_i(t, \cdot) - n_+\|_{L^1(\mathbb{R}_+)} = O\left(\frac{\sqrt{\log(1+t)}}{\sqrt{1+t}}\right)\,, \end{cases}$$

where n_+ is defined by (1.8). Moreover in the case of identical initial data $(n_1^0, \mu_1^0) = (n_2^0, \mu_2^0) = (n^0, \mu^0)$, we have a slightly better error estimate:

$$\|n(t, \cdot) - n_+\|_{L^1(\mathbb{R}_+)} = O\left(\frac{1}{\sqrt{1+t}}\right)\,.$$

On the other hand, for small initial conditions, or small parameter χ, we prove the self-similar decay of the solution. To state this result, we introduce the notation G for be the normalized Gaussian distribution on the half line

$$G(x) = \sqrt{\frac{2}{\pi}}\, e^{-\frac{x^2}{2}}\,, \quad x \ge 0.$$

Proposition 1.3 (Self similar decay in the subcritical case). *Let* $(n_i, \mu_i)_{1 \le i \le 2}$ *be the solution to* (1.2)–(1.3)–(1.4) *with initial data* $(n_i^0, \mu_i^0)_{1 \le i \le 2}$ *satisfying* (1.5). *Assume that*

- *either* $\chi < 4$,

- *or that* $\chi \ge 4$ *and that there exists* $\mu_0 \in (0, \mu_-)$ *such that for* $i = 1, 2$

$$(\forall x \ge 0) \quad N_i^0(x) \le N_{\mu_0}(x)\,.$$

Assume in addition that

$$\int_0^\infty n_i^0(x)(x^2 + \log n_i^0(x))\,\mathrm{d}x < +\infty\,.$$

Then, for $i = 1, 2$, *the following convergences hold true:*

$$\begin{cases} \mu_i(t) = O\left(\frac{1}{\sqrt{1+t}}\right)\,, \\ \left\|n_i(t, .) - \frac{1}{\sqrt{1+2t}} G\left(\frac{\cdot}{\sqrt{1+2t}}\right)\right\|_{L^1(\mathbb{R}_+)} = O\left(\frac{1}{(1+t)^{1/4}}\right)\,. \end{cases}$$

We shall refer to the zero state as the "silent state" where no dialog takes place between the cells. On the other hand, we shall refer to the state (n_+, μ_+) as the "dialog state", where the cells feel each other with a high fraction of activated protein at the cell boundary. Informally, our results state that, if $\chi > 4$, then the system is bistable: if the cells do not invest enough in the communication with their partner, they do not respond to each other, and their molecular content spreads out; but a sufficient

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$\chi < 4$					
Convergence towards the zero steady state (*no dialog*)					
$\chi > 4$					
μ	$0 < \mu < \mu_-$	μ_-	$\mu_- < \mu < \mu_+$	μ_+	$\mu_+ < \mu$
N_μ	supersolution	steady state	subsolution	steady state	supersolution
	If $N^0_{\{1,2\}}(x) < N_{\mu_-}(x)$, $n_{\{1,2\}} \to 0$ and $\mu_{\{1,2\}} \to 0$ *no dialog*	linearly unstable	If $N_{\mu_-}(x) < N^0_{\{1,2\}}(x)$, $n_{\{1,2\}} \to n_+$ and $\mu_{\{1,2\}} \to \mu_+$ *pairwise dialog*		
Bifurcation diagram					

Table 1: Informal summary of the results and sketch of the bifurcation diagram. We refer to the main text for the exact statements and assumptions.

initial input from each cell in the dialog leads to the establishment of a stable activated state. The two stable states are separated by an unstable state associated with a low level of activation. Results are summarized in Table 1. Notice that not all the initial states are contained. In particular, we could not deal with the case where one cell has a high input whereas its partner has a low input initially: $N^0_1(x) < N_{\mu_-} < N^0_2(x)$.

The article is organized as follows: Section 2 is devoted to the biological assumptions underlying our model. A useful comparison principle is stated in Section 3. Section 4 is devoted to the proof of the non-linear stability of the most concentrated steady state (Proposition 1.2). In Section 5, self-similar decay is establihed in the sub-critical case (Proposition 1.3). Linear stability analysis is postponed to the Appendix (Proposition 1.1). Some notations and mathematical useful inequalities are recalled in the last section of the Appendix.

2   Further biological background

Many events in plant and animal development depend on the ability of cells to interact with each other. Examples include the interaction of neuronal growth cones with target cells, cell division and differentiation, cell dissemination, inter-organ communications, tissue morphogenesis or regeneration.
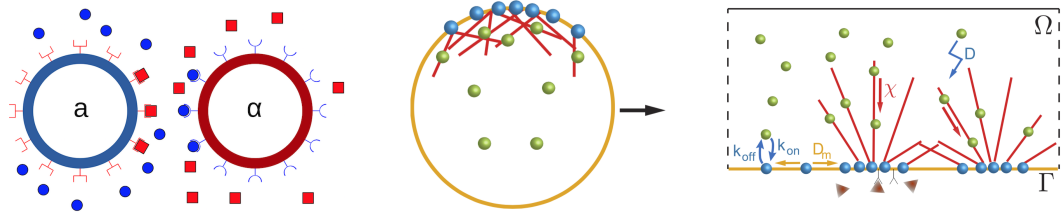
Figure 2: (Left) Yeast cells of both types (**a** or α) secrete some pheromone ($S_\mathbf{a}$ or S_α) and bear a pheromone receptor to detect the signal from the opposite cell type. (Right) A two-dimensional cartoon of protein dynamics inside a cell. Actin is polymerized into short filaments, that interact with each other. They are bundled together to form actin cables (which are part of the cytoskeleton) that cross the cell. In our model, the nucleation of filaments is influenced both by the concentration of Cdc42 at the membrane (the proteins that are actively transported along actin filaments) and by the concentration of external pheromone.

Understanding how cells dynamically integrate and respond to external signals from the microenvironment is a very challenging question. From the biological viewpoint, a generic model for cell communication is given by yeast cell mating. Yeast cell communication involves some intra-cellular proteins (as Cdc42), the cell cytoskeleton and extra-cellular pheromone molecules, Fig. 2 [1].

Several mathematical models have been proposed in the past decade. They incorporate many aspects of the molecular mechanisms involved in pheromone-induced protein aggreagtion. Although some of these models have been tested for their ability to fit quantitative data [17, 33, 19, 31, 13, 30, 14, 26, 23, 12], they have not been quantitatively assessed for their ability to make accurate predictions with no additional free parameter. In the model presented here, we focus on the intracellular actin-mediated transport of the protein Cdc42 which was shown to contribute to the initiation of cell polarization via inhomogeneous protein distribution at the cell surface [38, 39], but see [27, 33, 36] for a recent discussion about this hypothesis. Other factors, such as the Bem1-mediated positive feedback e.g., are not present in the model considered here. Indeed, we believe it is of interest to investigate the potentiality of a mass-preserving mechanism to account for symmetry breaking in the cell content.

The present work is the continuation of our analysis program about a model which was first introduced in [21], then studied in [9, 8] and finally tested for its ability to predict experimental data in [32] at the level of a single cell. This model relies on a coarse-grained description of actin-mediated transport and it is expressed by a non-linear and non-local partial differential equation. Here, we enrich the single cell model, and we push further our mathematical analysis in order to study pairwise cell-cell communication.

Although the present study is restricted to the one-dimensional setting, it is instructive to describe a higher dimensional situation, say the two-dimensional setting for the sake of simplicity. Denoting respectively by $n(t, x)$ the concentration of the Cdc42 protein and by $c(t, x)$ the density of actin filaments in the cytoplasm of the cell (the bounded domain $\Omega \subset \mathbb{R}^2$), and denoting by $\mu(t, s)$ the concentration of Cdc42 on the membrane of the cell ($\Gamma = \partial\Omega$), the model is:

$$\begin{cases} \partial_t n = \Delta n - \chi \nabla \cdot (n \nabla c)\,, & \text{in } \Omega\,, \\ \partial_t \mu = \partial_{ss} \mu + n - \mu\,, & \text{on } \Gamma\,, \end{cases} \tag{2.1}$$

where s is a parametrisation of the boundary Γ.

The advection term $\chi \, \nabla c$ accounts for active transport of proteins along actin filaments. Nucleation of new filaments is assumed to occur at the plasma membrane, under the combined action of Cdc42 and external pheromone molecules. After a dimensional analysis, the model that describes the actin filaments density is the following one:

$$\begin{cases} -\Delta \, c = 0 \,, & \text{in } \Omega \,, \\ -\nabla c \cdot \mathbf{e} = S\mu \,, & \text{on } \Gamma \,, \end{cases} \tag{2.2}$$

where $\mathbf{e}$ is the unit outward normal vector and $S(t, s)$ is the trace of the pheromone external signal on the cell membrane. Equations (2.1) and (2.2) are complemented by initial conditions and by an additional boundary condition on the cell membrane which guarantees the conservation of the total Cdc42 pool in the cell:

$$(\nabla n - \chi \, n \, \nabla c) \cdot \mathbf{e} = -\partial_t \mu \quad \text{on } \Gamma \,.$$

In the one dimensional case where the cytoplasm of the cell is modelled by the half line $\Omega = \mathbb{R}_+$, and the membrane is located at $x = 0$, the model (2.1)–(2.2) simply read as

$$\partial_t n = \partial_{xx} n + S\mu \, \partial_x n \,, \quad t > 0 \,, \quad x > 0 \,, \tag{2.3}$$

with an additionnal flux boundary condition that assures mass conservation. This latter equation has been analyzed in [9, 8, 28], its dynamics is well understood and is reminiscent of the Keller-Segel model in two dimensions. The principal result of [8] was to identify regimes in which non homogeneous stationary states, that were interpreted as polarised states, emerge.

In nature, the budding yeast, *Saccharomyces cerevisiae*, exists either in diploid state, or in haploid state with two possible types (**a** or α). Cells of both types secrete some pheromone ($S_{\mathbf{a}}$ or S_α), Fig. 2, and bear a pheromone receptor to detect the pheromone produced by the cells of the opposite type, [22]. In the present work we propose a model where the release of extra-cellular pheromone ($S_{\mathbf{a}}$ or S_α) depends on the concentration of the protein Cdc42 at the membrane [34]. Furthermore, according to the biological litterature, [29, 3, 35, 36], we assume that the pheromone contributes to the nucleation of new filaments at the plasma membrane of the cell of the opposite type, see Fig 2. To describe the protein dynamics on each cell membrane we use and enrich the model (2.3) in the one-dimensional case. This leads to a coupled system of two non-linear transport-diffusion equations set in two opposite half-lines. Throughout this work, the cell type is referred to by the subscript $i = 1, 2$.

This additional level of cellular pairwise communication leads to the following rectification of the model (2.3): for the cell i, we consider that the advection field is $-\chi S_j(t)\mu_i(t)$ where S_j is the concentration of pheromone produced by the cell of the opposite type j. Moreover, in this work we postulate that S_j is proportional to μ_j, which means that the pheromone produced by cell of type i is influenced by the level of activation of the opposite cell, possibly discounted by a damping factor depending on the inter-cellular distance (not explicitly written here). Thus, in both cells the advection field is the same: $-\chi\mu_1(t)\mu_2(t)$. From the biological point of view this model describes whether proteins Cdc42 will mainly get attached to the membrane, or will mainly diffusive in the bulk. The former event will be referred to as a "polarized state" for each cell, as in [9, 8, 32]. The case where both cells are polarized will be referred to as the "dialog state" between the two cells.

3   The comparison principle and its consequences

We start noticing that there is no direct comparison principle on (1.2)–(1.3)–(1.4). In this section, we first establish a concentration comparison principle on the cumulated distribution functions N_i (1.6),

reminiscent of [28], and analogous to the radially symmetric Keller-Segel system, see for instance [4, 24] and references therein. In a second step, we identify a family of sub/supersolutions that play a pivotal role in the non-linear stability analysis.

3.1 The concentration-comparison principle

For $i = 1, 2$, the integrated quantities $N_i(t, x) = \mu_i(t) + \int_0^x n_i(t, y) \, \mathrm{d}y$ associated with (1.2)–(1.3)–(1.4) and (1.5) satisfy

$$
\begin{cases}
\partial_t N_i(t, x) - \partial_{xx} N_i(t, x) - \chi \mu_1(t) \mu_2(t) \partial_x N_i(t, x) = 0 \,, \\
N_i(t, 0) = \mu_i(t) \,, \qquad \lim_{x \to \infty} N_i(t, x) = 1 \,, \\
\frac{\mathrm{d}\mu_i}{\mathrm{d}t}(t) = \partial_x N_i(t, 0) - \mu_i(t) \,.
\end{cases}
\tag{3.1}
$$

Definition 3.1. *A supersolution (resp. subsolution) to (3.1) is a couple of nondecreasing functions (with respect to x) $(\overline{N}_1, \overline{N}_2)$ (resp. $(\underline{N}_1, \underline{N}_2)$) satisfying*

$$
\begin{cases}
\partial_t \overline{N}_i(t, x) - \partial_{xx} \overline{N}_i(t, x) - \chi \overline{\mu}_1(t) \overline{\mu}_2(t) \partial_x \overline{N}_i(t, x) \geq 0 \,, \\
\overline{N}_i(t, 0) = \overline{\mu}_i(t) \,, \qquad \lim_{x \to \infty} \overline{N}_i(t, x) = 1 \,, \\
\frac{\mathrm{d}\overline{\mu}_i}{\mathrm{d}t}(t) \geq \partial_x \overline{N}_i(t, 0) - \overline{\mu}_i(t) \,,
\end{cases}
$$

with similar definition for a subsolution by changing $\geq$ into $\leq$.

We now state the concentration comparison principle.

Lemma 3.2 (Comparison principle). *Let $(\overline{N}_1, \overline{N}_2)$ and $(\underline{N}_1, \underline{N}_2)$ be respectively smooth super and subsolution to (3.1) defined on $[0, T] \times \mathbb{R}_+$. Assume that for $i = 1, 2$*

$$
(\forall x \geq 0) \quad \underline{N}_i(0, x) \leq \overline{N}_i(0, x) \,, \quad and \quad \overline{\mu}_i(0) > \underline{\mu}_i(0) \,.
$$

Then, the following inequalities hold true for $i = 1, 2$,

$$
(\forall t \in (0, T)) \, (\forall x \geq 0) \quad \underline{N}_i(t, x) \leq \overline{N}_i(t, x) \,.
$$

Proof. For $i = 1, 2$, denoting $F_i = \overline{N}_i - \underline{N}_i$, one has

$$
\begin{cases}
\partial_t F_i - \partial_{xx} F_i - \chi \overline{\mu}_1(t) \overline{\mu}_2(t) \partial_x F_i \geq \chi \left(\overline{\mu}_1(t) \overline{\mu}_2(t) - \underline{\mu}_1(t) \underline{\mu}_2(t) \right) \partial_x \underline{N}_i \,, \\
F_i(t, 0) = \overline{\mu}_i(t) - \underline{\mu}_i(t) \,, \qquad \lim_{x \to \infty} F_i(t, x) = 0 \,, \\
\frac{\mathrm{d}}{\mathrm{d}t} \left(\overline{\mu}_i(t) - \underline{\mu}_i(t) \right) \geq \partial_x F_i(t, 0) - (\overline{\mu}_i(t) - \underline{\mu}_i(t)) \,, \\
F_i(0, x) = \overline{N}_i(0, x) - \underline{N}_i(0, x) \geq 0 \,.
\end{cases}
$$

The bootstrap acts as follows. Since we are dealing with nondecreasing functions N_i, the quantity $\chi \left(\overline{\mu}_1(t) \overline{\mu}_2(t) - \underline{\mu}_1(t) \underline{\mu}_2(t) \right) \partial_x \underline{N}_i$ is nonnegative as long as

$$
\overline{\mu}_1(t) \overline{\mu}_2(t) - \underline{\mu}_1(t) \underline{\mu}_2(t) \geq 0 \,,
$$

holds true. This is in particular the case if

$$
\overline{\mu}_1(t) \geq \underline{\mu}_1(t) \quad and \quad \overline{\mu}_2(t) \geq \underline{\mu}_2(t) \,.
\tag{3.2}
$$

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Recalling the assumption $\overline{\mu}_i(0) > \underline{\mu}_i(0)$, we denote by $\tau > 0$ the first time for which an equality in (3.2) occurs. Suppose without loss of generality that $\overline{\mu}_1(\tau) = \underline{\mu}_1(\tau)$. Let us define the function $z(t)$ by

$$z(t) := \overline{\mu}_1(t) - \underline{\mu}_1(t).$$

Then, for all $(t, x) \in (0, \tau) \times (0, \infty)$, one has

$$\begin{cases} \partial_t F_1(t, x) - \partial_{xx} F_1(t, x) - \chi \overline{\mu}_1(t) \overline{\mu}_2(t) \partial_x F_1(t, x) \geq 0, \\ F_1(t, 0) = \overline{\mu}_1(t) - \underline{\mu}_1(t) = z(t) \geq 0, \qquad \lim_{x \to \infty} F_1(t, x) = 0, \\ \frac{dz}{dt}(t) \geq \partial_x F_1(t, 0) - z(t). \end{cases}$$

Recalling in addition the assumption $F_i(0, 0) > 0$, it follows that there exists a nonnegative compactly supported (in $(0, +\infty)$) function $f \neq 0$ such that $f(x) \leq F_1(0, x)$ and $f(0) = 0$. Next, we consider the solution to the parabolic equation

$$\begin{cases} \partial_t g(t, x) - \partial_{xx} g(t, x) - \chi \overline{\mu}_1(t) \overline{\mu}_2(t) \partial_x g(t, x) = 0, \\ g(t, 0) = 0, \\ g(0, x) = f(x). \end{cases}$$

On the one hand, it follows from the standard maximum principle [18] that $F_1 \geq g$ on $[0, \tau] \times [0, \infty)$. On the other hand, classical parabolic regularity implies that $g(t, x) > 0$ for all $(t, x) \in (0, \tau] \times (0, \infty)$. Hence, applying the Hopf Lemma [18], it follows that $\partial_x g(\tau, 0) > 0$. Next, from the equality $F_1(\tau, 0) = g(\tau, 0) = 0$, we deduce that $\partial_x F_1(\tau, 0) \geq \partial_x g(\tau, 0) > 0$. Consequently one has $\frac{d}{dt} z(\tau) > 0$ which contradicts $z > 0$ on $[0, \tau)$ and $z(\tau) = 0$. $\qquad \square$

3.2 Remarkable sub/supersolutions

For any parameter $\mu \in (0, 1)$, the pair (N_μ, N_μ) is either a subsolution or a supersolution, depending on the sign of $P(\mu)$ with P defined by (1.7) and $N_\mu(x) = \mu + (1 - \mu)\left(1 - e^{-\chi \mu^2 x}\right)$.

Lemma 3.3. *The pair (N_μ, N_μ) where*

$$N_\mu(x) = \mu + (1 - \mu)(1 - e^{-\chi \mu^2 x}),$$

is a supersolution (resp. a subsolution) to (3.1) *if*

$$P(\mu) = \chi \mu^2 - \chi \mu + 1 \geq 0 \quad (resp. \quad \leq 0).$$

Recall that $P(\mu) \leq 0$, hence (N_μ, N_μ) is a subsolution, if and only if $\mu \in [\mu_-, \mu_+]$. However the interval is empty if $\chi < 4$. Indeed, in the latter case, the pair (N_μ, N_μ) is always a supersolution. The various cases are reported in Table 1.

Proof. A straightforward computation yields the following identities:

$$\begin{cases} \partial_t N_\mu - \partial_{xx} N_\mu - \chi \mu^2 \partial_x N_\mu = 0, \\ N_\mu(0) = \mu, \\ \partial_x N_\mu(0) - \mu = (1 - \mu)\chi \mu^2 - \mu = -\mu \left(\chi \mu^2 - \chi \mu + 1\right). \end{cases}$$

The conclusion follows from the definition of P (1.7). $\qquad \square$

4   Non-linear stability of (n_+, μ_+): proof of Proposition 1.2

We split the proof in several steps. We begin with the symmetric case where the cells are identical: $(n_1, \mu_1) = (n_2, \mu_2)$. Then, the result is extended to the general case by using the comparison principle.

4.1   The symmetric case

In this part, we assume that $(n_1^0, \mu_1^0) = (n_2^0, \mu_2^0) = (n^0, \mu^0)$, so that the solutions (n_1, μ_1) and $(n_2, \mu_2) = (n, \mu)$ are always the same. We keep all the assumptions, in particular (1.5), i.e. $M_i = 1$. We drop the subscript $i = 1, 2$ for the sake of clarity. Our goal is to prove Proposition 1.2 in the symmetric case.

$$\begin{cases} \partial_t n(t, x) - \partial_{xx} n(t, x) - \chi \mu(t)^2 \partial_x n(t, x) = 0, \quad (t, x) \in (0, \infty)^2, \\ \partial_x n(t, 0) + \chi \mu(t)^2 n(t, 0) = \frac{\mathrm{d}\mu}{\mathrm{d}t}(t) = n(t, 0) - \mu(t). \end{cases} \tag{4.1}$$

The main argument is based on the construction of a suitable Lyapunov function, which is a mixture of entropy and quadratic scalar contributions, similarly as for the case of a single cell [28], see Lemma 4.1 below. Next, the assumptions made on the initial condition allow to obtain lower bounds on the terms involved in entropy dissipation and to prove that the Lyapunov functional tends to zero. The keystone is to establish that μ stays away from μ_- in order to control the relaxation of $\mu(t)$ towards μ_+.

We define the Lyapunov functional L by:

$$L(t) := (1 - \mu(t)) \mathcal{H}(m(t, \cdot) | m_+) = \int_0^\infty n(t, x) \log \frac{m(t, x)}{m_+(x)} \, \mathrm{d}x \geq 0,$$

where m_+ is given by (1.8), $\mathcal{H}(n|p)$ is the relative entropy [2], and

$$m(t, x) = \frac{n(t, x)}{\int_0^\infty n(t, x) \, \mathrm{d}x} = \frac{n(t, x)}{1 - \mu(t)},$$

is the renormalized density in the bulk. In addition, the function f is defined by

$$f(\mu) = \mu \log \frac{\mu}{\mu_+} + (1 - \mu) \log \frac{1 - \mu}{1 - \mu_+} + \chi \frac{(\mu - \mu_+)^2 (\mu + 2\mu_+)}{3}.$$

The function f is designed such as $f(\mu_+) = 0$, and

$$f'(\mu) = \log \frac{\mu(1 - \mu_+)}{\mu_+(1 - \mu)} + \chi \left(\mu^2 - \mu_+^2 \right). \tag{4.2}$$

In particular, the function f is nonincreasing on $(0, \mu_+)$ and nondecreasing on $(\mu_+, 1)$, hence it is nonnegative on $(0, 1)$.

Lemma 4.1. *Let (n, μ) be the solution to* (4.1). *The following inequality holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(L(t) + f(\mu(t)) \right) \leq -D(t)^2 - \chi \left(\mu(t)^2 - \mu_+^2 \right) \mu(t) P(\mu(t)), \tag{4.3}$$

where D^2 is defined as

$$D^2 = \int_0^\infty n \left(\partial_x \log n + \chi \mu^2 \right)^2 \, \mathrm{d}x = (1 - \mu) \mathcal{I} \left(m | \chi \mu^2 e^{-\chi \mu^2 x} \right),$$

and $\mathcal{I}(p|q) = \int_0^\infty p \left(\partial_x \log \frac{p}{q} \right)^2 \, \mathrm{d}x$ is the Fisher information.

Proof of Lemma 4.1. The Lyapunov functional L can be rearranged as follows,

$$L = \int_0^\infty \left(n \log n - n \log m_+ - n \log(1-\mu) \right) \, \mathrm{d}x$$

$$= \int_0^\infty \left(n \log n - n \log m_+ \right) \, \mathrm{d}x - (1-\mu)\log(1-\mu) \,.$$

Hence, differentiating L, we obtain

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \int_0^\infty \partial_t n \left(1 + \log n - \log m_+ \right) \, \mathrm{d}x + (1 + \log(1-\mu)) \frac{\mathrm{d}\mu}{\mathrm{d}t} \,,$$

and using that $\int_0^\infty n \, \mathrm{d}x + \mu = 1$, we get

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \int_0^\infty \partial_t n \log \frac{n}{m_+} \, \mathrm{d}x + \log(1-\mu) \frac{\mathrm{d}\mu}{\mathrm{d}t} \,.$$

Next, using the definition of m_+ (1.8), that (n,μ) is solution to (4.1), an integration by parts yields the following identities:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \int_0^\infty \log \frac{n}{m_+} \left(\partial_{xx} n + \chi \mu^2 \partial_x n \right) \, \mathrm{d}x + \log(1-\mu) \frac{\mathrm{d}\mu}{\mathrm{d}t}$$

$$= -\left(\log \frac{n(t,0)}{m_+(0)} \right) \frac{\mathrm{d}\mu}{\mathrm{d}t} - \int_0^\infty n (\partial_x \log n + \chi \mu_+^2)(\partial_x \log n + \chi \mu^2) \, \mathrm{d}x + \log(1-\mu) \frac{\mathrm{d}\mu}{\mathrm{d}t}$$

$$= -\left(\log \frac{n(t,0)}{(1-\mu)m_+(0)} \right) \frac{\mathrm{d}}{\mathrm{d}t}\mu - \underbrace{\int_0^\infty n (\partial_x \log n + \chi \mu^2)^2 \, \mathrm{d}x}_{=D^2} + \chi(\mu^2 - \mu_+^2) \int_0^\infty (\partial_x n + \chi \mu^2 n) \, \mathrm{d}x$$

$$= -D^2 - \left(\log \frac{n(t,0)}{(1-\mu)m_+(0)} \right) \frac{\mathrm{d}\mu}{\mathrm{d}t} + \chi(\mu^2 - \mu_+^2)(-n(t,0) + \chi \mu^2 (1-\mu)) \,.$$

Using now that $\frac{\mathrm{d}}{\mathrm{d}t}\mu = n(t,0) - \mu$, we deduce that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -D^2 + \left(\log \frac{\mu}{n(t,0)} + \log \frac{(1-\mu)m_+(0)}{\mu} \right) \frac{\mathrm{d}\mu}{\mathrm{d}t} + \chi(\mu^2 - \mu_+^2)\left(-\frac{\mathrm{d}\mu}{\mathrm{d}t} - \mu + \chi \mu^2 (1-\mu) \right)$$

$$= -D^2 + \underbrace{(n(t,0) - \mu) \log \frac{\mu}{n(t,0)}}_{\leq 0} + \chi(\mu^2 - \mu_+^2)\mu \left(-\chi\mu^2 + \chi\mu - 1 \right)$$

$$+ \left(\chi\mu_+^2 - \chi\mu^2 + \log \frac{(1-\mu)\chi\mu_+^2}{\mu} \right) \frac{\mathrm{d}\mu}{\mathrm{d}t} \,.$$

The conclusion follows by using the definitions of f (4.2) and P (1.7), which in turn implies $\chi\mu_+^2 = \frac{\mu_+}{1-\mu_+}$. $\qquad\square$

An important observation is that, for all $\mu \in [\mu_-, 1]$, the quantity $(\mu^2 - \mu_+^2)P(\mu)$ is nonnegative. As a consequence, the second contribution in the r.h.s. of (4.3) is dissipative provided $\mu(t)$ remains above μ_-. Such an *a priori* control enables to prove that $L + f$ converges to zero with an algebraic rate.

Lemma 4.2. *Under the same assumptions as in Proposition 1.2, there exists a constant C depending on $(\underline{\mu}_0, \overline{\mu}_0)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L(t) + f(\mu(t)) \right) \leq -C\left(L(t) + f(\mu(t)) \right)^2 \,.$$

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Proof of Lemma 4.2. We first note that $N_{\underline{\mu}_0}(x)$ is a subsolution, whereas $N_{\overline{\mu}_0}(x)$ is a supersolution, see Lemma 3.3 and (1.9) (see also Table 1). As the initial data are well ordered by assumption (1.10), we deduce from the comparison principle in Lemma 3.2 that

$$\forall (t,x) \in (0,\infty) \times \mathbb{R}_+ , \quad N_{\underline{\mu}_0}(x) \le N(t,x) \le N_{\overline{\mu}_0}(x) . \tag{4.4}$$

In particular, we have $\mu(t) \ge \underline{\mu}_0 > \mu_-$. Hence, $(\mu(t)^2 - \mu_+^2)P(\mu(t))$ is nonnegative. Furthermore, the family of functions N_μ is increasing with respect to μ, thus

$$N_{\underline{\mu}_0}(x) \le \mu(t) + (1 - \mu(t))\left(1 - e^{-\chi\mu(t)^2 x}\right) \le N_{\overline{\mu}_0}(x) . \tag{4.5}$$

Using the quadratic Wasserstein distance W, whose definition is recalled in the Appendix, together with the definition of $m(t,x) = \frac{n(t,x)}{\int_0^\infty n(t,x)\,\mathrm{d}x} = \frac{n(t,x)}{1-\mu(t)}$, we see that

$$W\left(m, \chi\mu^2 e^{-\chi\mu^2 x}\right) = \frac{1}{\sqrt{1-\mu}} W\left(\mu\delta_0 + (1-\mu)m, \mu\delta_0 + (1-\mu)\chi\mu^2 e^{-\chi\mu^2 x}\right) ,$$

where δ_0 is the Dirac mass in $x = 0$. The bounds on N and N_μ in (4.4) and (4.5), allow to apply lemma B.3 to obtain

$$W\left(\mu\delta_0 + (1-\mu)m, \mu\delta_0 + (1-\mu)\chi\mu^2 e^{-\chi\mu^2 x}\right) \le C_W(\underline{\mu}_0, \overline{\mu}_0) ,$$

where

$$C_W(\underline{\mu}_0, \overline{\mu}_0) = W\left(\underline{\mu}_0\delta_0 + (1-\underline{\mu}_0)\chi\underline{\mu}_0^2 e^{-\chi\underline{\mu}_0^2 x}, \overline{\mu}_0\delta_0 + (1-\overline{\mu}_0)\chi\overline{\mu}_0^2 e^{-\chi\overline{\mu}_0^2 x}\right) < +\infty .$$

Then, applying the HWI inequality, see Lemma B.6 in the Appendix, to the exponential measure $\chi\mu^2 e^{-\chi\mu^2 x}$, we obtain that

$$D^2 = (1-\mu)\,\mathcal{I}\left(m|\chi\mu^2 e^{-\chi\mu^2 x}\right) \ge (1-\mu)\left(\frac{\mathcal{H}\left(m|\chi\mu^2 e^{-\chi\mu^2 x}\right)}{W\left(m, \chi\mu^2 e^{-\chi\mu^2 x}\right)}\right)^2$$

$$\ge (1-\mu)^2\left(\frac{\mathcal{H}\left(m|\chi\mu^2 e^{-\chi\mu^2 x}\right)}{C_W(\underline{\mu}_0, \overline{\mu}_0)}\right)^2 . \tag{4.6}$$

Recalling the definition of the relative entropy, $\mathcal{H}(n|p) = \int_0^\infty n \log(n/p)\,\mathrm{d}x$, we see that

$$\mathcal{H}\left(m|\chi\mu^2 e^{-\chi\mu^2 x}\right) = \frac{1}{1-\mu}\left(\underbrace{(1-\mu)\mathcal{H}(m|m_+)}_{=L} + \Delta(\mu)\right) ,$$

where the correction $\Delta(\mu)$ is defined by

$$\Delta(\mu) = (1-\mu)\left(\log\frac{\mu_+^2}{\mu^2}\right)\int_0^\infty m\,\mathrm{d}x + \chi(1-\mu)(\mu^2 - \mu_+^2)\int_0^\infty xm\,\mathrm{d}x .$$

We deduce from (4.3) and (4.6) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(L + f(\mu)) \le -\frac{(L + \Delta(\mu))^2}{(1-\mu)C_W(\underline{\mu}_0, \overline{\mu}_0)^2} - \chi(\mu^2 - \mu_+^2)\mu P(\mu) \le 0 .$$

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Noticing additionally that

$$(L + f)^2 = (L + \Delta)^2 + (f - \Delta)^2 + 2(L + \Delta)(f - \Delta) \leq \left(1 + \frac{1}{\alpha}\right)(L + \Delta)^2 + (1 + \alpha)(f - \Delta)^2,$$

we see that for any $\alpha > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(L + f(\mu)) \leq \frac{-1}{(1 - \mu)C_W(\underline{\mu}_0, \overline{\mu}_0)^2}\left(\frac{\alpha}{1 + \alpha}(L + f(\mu))^2 - \alpha(f(\mu) - \Delta(\mu))^2 + R(\mu)\right),$$

where $R(\mu) = \chi C_W(\underline{\mu}_0, \overline{\mu}_0)^2 \mu(1 - \mu)(\mu^2 - \mu_+^2)P(\mu)$.

It is useful to notice that $\int xm\,\mathrm{d}x$ is uniformly bounded, as it follows from the estimate (4.4) together with an integration by parts:

$$\int_0^\infty xm(t, x)\,\mathrm{d}x = \frac{1}{1 - \mu}\int_0^\infty (1 - N(t, x))\,\mathrm{d}x \leq \frac{1}{1 - \overline{\mu}_0}\int_0^\infty \left(1 - N_{\underline{\mu}_0}\right)\,\mathrm{d}x = \frac{1 - \underline{\mu}_0}{\chi\underline{\mu}_0^2(1 - \overline{\mu}_0)}.$$
(4.7)

Therefore, both $|\Delta|$ and $|f|$ are $O(|\mu - \mu_+|)$ in $(\underline{\mu}_0, 1)$. we deduce that $(f(\mu) - \Delta(\mu))^2 = O(R(\mu))$ in a neighbourhood of μ_+, as μ_+ is a root of P. Hence, there exists a constant $\alpha_0 > 0$ such that

$$(\forall \mu \in (\underline{\mu}_0, 1)) \quad -\alpha_0(f(\mu) - \Delta(\mu))^2 + R(\mu) \geq 0.$$

Choosing $C = \frac{\alpha_0}{(1 + \alpha_0)(1 - \underline{\mu}_0)C_W(\underline{\mu}_0, \overline{\mu}_0)^2}$, this achieves the proof of Lemma 4.2. □

Corollary 4.3. *Under the same assumptions as in Proposition 1.2, the following estimates hold*

$$(\forall t > 0), \quad 0 \leq \int_0^\infty n(t, x)\log\frac{m(t, x)}{m_+(x)}\,\mathrm{d}x + (\mu(t) - \mu_+)^2 \leq \frac{C}{1 + t},$$
(4.8)

and

$$\forall t > 0, \quad \|n - n_+\|_{L^1(\mathbb{R}_+)} \leq \frac{C}{\sqrt{1 + t}},$$
(4.9)

where $n_+ = (1 - \mu_+)m_+$ (1.8).

Proof. The first inequality is a direct consequence of lemma 4.2. The second one is obtained by the triangle inequality:

$$\|n - (1 - \mu_+)m_+\|_{L^1(\mathbb{R}_+)} \leq (1 - \mu)\|m - m_+\|_{L^1(\mathbb{R}_+)} + |\mu - \mu_+|,$$

We apply finally the Csiszar-Kullbáck inequality (see proposition B.7):

$$\|m - m_+\|_{L^1(\mathbb{R}_+)} \leq \sqrt{2\mathcal{H}(m|m_+)} \leq \sqrt{\frac{2L(t)}{1 - \mu(t)}},$$

from which we can deduce (4.9), based on (4.8). □

4.2 Refined estimates of convergence for special solutions

On the way to extending the previous result to the general, non symmetric case, further estimates of convergence are required. Below, we focus on two families of solutions, those which are initialized with $N_{\underline{\mu}_0}(x)$, resp. $N_{\overline{\mu}_0}(x)$, where $\underline{\mu}_0$ and $\overline{\mu}_0$ are as in Proposition 1.2. This is of particular interest as any solution of the general problem with suitable initial data fits in between these two solutions, due to the comparison principle 3.2. For the sake of brevity, we denote by $(\underline{\mu}, \underline{n})$, resp. $(\overline{\mu}, \overline{n})$ the solution of the symmetric problem with the initial data corresponding to $N_{\underline{\mu}_0}(x)$, resp. $N_{\overline{\mu}_0}(x)$.

Corollary 4.3 asserts that the error $\mu(t) - \mu_+$ is of order $(1 + t)^{-1/2}$ at most. However, it is possible to gain a better rate of convergence for $(\overline{\mu}, \overline{n})$, in integral form, or alternatively in the Wasserstein distance W_1 (which accounts for a weaker convergence than the L^1 norm).

Lemma 4.4. *Under the same assumptions as in Proposition 1.2, the error $(\overline{\mu} - \mu_+)$ belongs to $L^1(\mathbb{R}_+)$. Furthermore, there exist a constant C, such that the following estimates hold true*

$$\int_t^\infty |\overline{\mu}(s) - \mu_+| \, \mathrm{d}s \leq \frac{C}{\sqrt{1+t}}, \tag{4.10}$$

$$W_1(\overline{\mu}\delta_0 + \overline{n}, \mu_+\delta_0 + n_+) \leq \frac{C}{\sqrt{1+t}}. \tag{4.11}$$

$$|\overline{\mu} - \mu_+| \leq \frac{C}{\sqrt{1+t}}. \tag{4.12}$$

Similar results hold for $(\underline{\mu}, \underline{n})$.

Proof. The last inequality (4.12) follows from (4.8) which was established in corollary 4.3. Let us start with the first point of the Lemma. Since $\overline{\mu} - \mu_+ \geq 0$, the absolute value can be removed in (4.10). Recall that the Wasserstein distance W_1 involves the L^1 norm of the cumulated distribution function: if we define $\overline{N}$ and N_+ as in (1.6) for $(\overline{\mu}, \overline{n})$ and (μ_+, n_+) respectively, one has

$$W_1(\overline{\mu}\delta_0 + \overline{n}|\mu_+\delta_0 + n_+) = \int_0^\infty |\overline{N} - N_+| \, \mathrm{d}x = \int_0^\infty \left(\overline{N} - N_+\right) \, \mathrm{d}x$$
$$= \int_0^\infty x \left(n_+ - \overline{n}\right) \, \mathrm{d}x, \tag{4.13}$$

recalling that n_+ does not depend on time, we deduce that,

$$\frac{\mathrm{d}}{\mathrm{d}t} W_1(\overline{\mu}\delta_0 + \overline{n}|\mu_+\delta_0 + n_+) = -\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty x\overline{n} \, \mathrm{d}x$$
$$= \int_0^\infty (\partial_x \overline{n} + \chi\overline{\mu}^2 n) \, \mathrm{d}x$$
$$= -\overline{n}(t,0) + \chi\overline{\mu}^2(1 - \overline{\mu})$$
$$= -\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mu} - \overline{\mu}P(\overline{\mu}),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{\mu} - \mu_+ + W_1(\overline{\mu}\delta_0 + \overline{n}|\mu_+\delta_0 + n_+)\right) = -\overline{\mu}P(\overline{\mu})$$
$$\leq -\mu_+ P'(\mu_+)(\overline{\mu} - \mu_+) \leq 0, \tag{4.14}$$

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by the convexity of P and $P(\mu_+) = 0$. In particular, we deduce that

$$\int_0^\infty |\overline{\mu}(t) - \mu_+| \, \mathrm{d}t \leq \frac{1}{\mu_+ P'(\mu_+)} \left(\int_0^\infty |N_{\overline{\mu}_0}(x) - N_+(x)| \, \mathrm{d}x + \overline{\mu}_0 - \mu_+ \right) < +\infty \, ,$$

which is the first statement in the Lemma. In order to proceed with the latter estimates (4.10)–(4.11), we shall re-use the entropy part of the error estimate (4.8). Indeed, this provides some information about the first order moments, as stated in the following intermediate result.

Lemma 4.5. *Let p be a probability distribution on $\mathbb{R}_+$ such that for some $\lambda > 0$, the following relative entropy is finite:*

$$\mathcal{H}\left(p | \lambda e^{-\lambda x}\right) = \int_0^\infty p \log \frac{p}{\lambda e^{-\lambda x}} \, \mathrm{d}x < +\infty \, .$$

Denote $J = \int_0^\infty x p \, \mathrm{d}x$. It is finite and is controlled as follows:

$$\mathcal{H}\left(p | \lambda e^{-\lambda x}\right) \geq \mathcal{H}\left(p | \frac{1}{J} e^{-\frac{x}{J}}\right) + \frac{\left(J - \lambda^{-1}\right)^2}{2\left(\lambda^{-2} + J^2\right)} \, .$$

Proof of Lemma 4.5. A straightforward computation gives

$$\mathcal{H}\left(p | \lambda e^{-\lambda x}\right) = \mathcal{H}\left(p | \frac{1}{J} e^{-\frac{x}{J}}\right) + \log \frac{1}{J\lambda} + \left(\lambda - \frac{1}{J}\right) \int_0^\infty x p \, \mathrm{d}x$$

$$\geq \mathcal{H}\left(p | \frac{1}{J} e^{-\frac{x}{J}}\right) + \lambda J - 1 - \log\left(\lambda J\right) \, ,$$

and the result follows from a Taylor expansion

$$z - 1 - \log z \geq \frac{(z-1)^2}{2 \max(1, z^2)} \geq \frac{(z-1)^2}{2(1+z^2)} \, .$$

$\square$

Keeping the notation $\overline{m} = (1 - \overline{\mu})\overline{n}$, as a direct application of Lemma 4.5 and (4.8), together with the *a priori* bound (4.7), we obtain that there is a constant C such that

$$\left(\int_0^\infty x \overline{m} \, \mathrm{d}x - \frac{1}{\chi \mu_+^2} \right)^2 \leq C \int_0^\infty \overline{n} \log \frac{\overline{m}}{m_+} \, \mathrm{d}x \leq \frac{C}{1+t} \, .$$

This enables to improve the previous estimation (4.13):

$$W_1(\overline{\mu}\delta_0 + \overline{n} | \mu_+ \delta_0 + n_+) \leq (1 - \mu_+) \int_0^\infty x m_+ \, \mathrm{d}x - (1 - \overline{\mu}) \int_0^\infty x \overline{m} \, \mathrm{d}x$$

$$\leq (\overline{\mu} - \mu_+) \int_0^\infty x m_+ \, \mathrm{d}x + (1 - \overline{\mu}) \left(\int_0^\infty x m_+ \, \mathrm{d}x - \int_0^\infty x \overline{m} \, \mathrm{d}x \right)$$

$$\leq \frac{C}{\sqrt{1+t}} \, ,$$

where we have also used the error estimate $|\overline{\mu} - \mu_+| = O\left((1+t)^{-1/2}\right)$. Finally, integrating (4.14) between t and $+\infty$, it follows that

$$\mu_+ P'(\mu_+) \int_t^\infty (\overline{\mu} - \mu_+) \, \mathrm{d}s \leq W_1(\overline{\mu}\delta_0 + \overline{n} | \mu_+ \delta_0 + n_+) + \overline{\mu} - \mu_+ \leq \frac{C}{\sqrt{1+t}} \, ,$$

which ends the proof of the Lemma 4.4.

$\square$

Corollary 4.6. *Under the same assumptions as in Proposition 1.2, one has*

$$\forall p \geq 1, \quad \int_t^\infty |\overline{\mu}(s) - \underline{\mu}(s)|^p \, \mathrm{d}s \leq \frac{C_p}{(1+t)^{p/2}},$$

and

$$\int_0^t (1+s)|\overline{\mu}(s) - \underline{\mu}(s)|^2 \, \mathrm{d}s \leq C(1 + \log(1+t)). \tag{4.15}$$

Proof of Corollary 4.6. From corollary 4.3 we already have $\sup_{s \geq t} |\overline{\mu}(s) - \underline{\mu}(s)|^2 \leq \frac{C}{1+t}$. We compute

$$\int_t^\infty |\overline{\mu} - \underline{\mu}|^p \, \mathrm{d}s \leq \sup_{s \geq t} |\overline{\mu} - \underline{\mu}|^{p-1} \int_t^\infty |\overline{\mu} - \underline{\mu}| \, \mathrm{d}s \leq \frac{C_p}{(1+t)^{p/2}}.$$

Additionnaly, we have

$$\int_0^t (1+s)|\overline{\mu} - \underline{\mu}|^2 \, \mathrm{d}s = \left[-(1+s) \int_s^\infty |\overline{\mu} - \underline{\mu}|^2 \, \mathrm{d}y \right]_0^t + \int_0^t \int_s^\infty |\overline{\mu} - \underline{\mu}|^2 \, \mathrm{d}y \, \mathrm{d}s$$

$$\leq C + C \int_0^t \frac{\mathrm{d}s}{1+s},$$

leading to the result. $\qquad\square$

4.3 The nonsymmetric case

We are now ready to prove Proposition 1.2 without the symmetry hypothesis. As a direct application of the comparison principle, $N(t, \cdot) \to N_+$ in L^1 norm (*i.e.* in W_1 distance between n and n_+). This can be reinforced to the convergence $n(t, \cdot) \to n_+$ in L^1 norm using the relative entropy, as in the symmetric problem.

As before, we introduce the Lyapunov functional L_i and the dissipation D_i by

$$L_i(t) = (1 - \mu_i)\mathcal{H}(m_i|m_+), \quad D_i^2 = (1 - \mu_i)\mathcal{I}(m_i|\chi\mu_1\mu_2 e^{-\chi\mu_1\mu_2 x}),$$

Performing similar computations as in Section 4.1, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(L_1 + L_2) = \sum_{i=1,2} \left(-D_i^2 + \left(\log \frac{\mu_i}{n_i(t,0)} \right) \frac{\mathrm{d}\mu_i}{\mathrm{d}t} + \left(\log \frac{(1-\mu_i)m_+(0)}{\mu_i} \right) \frac{\mathrm{d}\mu_i}{\mathrm{d}t} \right.$$

$$\left. + \chi \left(\mu_1\mu_2 - \mu_+^2 \right) \left(-n_i(t,0) + \chi\mu_1\mu_2(1-\mu_i) \right) \right). \tag{4.16}$$

Let the function $f_{\log}$ be defined by

$$f_{\log}(\mu_+) = 0, \qquad f'_{\log}(\mu) = \log \frac{\mu}{(1-\mu)m_+(0)} = \log \frac{\mu(1-\mu_+)}{(1-\mu)\mu_+}.$$

We observe that $f'_{\log}(\mu_+) = 0$ and the second derivative satisfies $f''_{\log}(\mu) > 0$ for $\mu \in (0,1)$. Hence, locally in the neighborhood of μ_+, the function $f_{\log}$ behaves as $(\mu - \mu_+)^2$.

Following (4.16), using that $\mu_1\mu_2 = \left(\frac{\mu_1+\mu_2}{2} \right)^2 - \left(\frac{\mu_1-\mu_2}{2} \right)^2$ and recalling that $\frac{\mathrm{d}}{\mathrm{d}t}\mu_i = n_i(t,0) - \mu_i$, we obtain

$$\chi(\mu_1\mu_2 - \mu_+^2)(n_1(t,0) + n_2(t,0)) = 2\frac{\mathrm{d}}{\mathrm{d}t}\left(g\left(\frac{\mu_1+\mu_2}{2} \right) \right)$$

$$+ \chi\left(\left(\frac{\mu_1+\mu_2}{2} \right)^2 - \mu_+^2 \right)(\mu_1 + \mu_2) - \chi\frac{(\mu_1-\mu_2)^2}{4}(n_1(t,0) + n_2(t,0)),$$

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where the function g is defined by

$$g(\mu) = \chi \left(\frac{\mu^3}{3} - \mu_+^2 \mu + 2\frac{\mu_+^3}{3} \right) = \chi \frac{(\mu - \mu_+)^2 (\mu + 2\mu_+)}{3}, \quad g'(\mu) = \chi \left(\mu^2 - \mu_+^2 \right).$$

We combine these contributions into the following functional $\widetilde{L}$:

$$\widetilde{L}(t) = L_1(t) + L_2(t) + f_{\log}(\mu_1(t)) + f_{\log}(\mu_2(t)) + 2g \left(\frac{\mu_1(t) + \mu_2(t)}{2} \right).$$

From the comparison principle and the HWI inequality, it follows that there exists a positive constant C such that $D_1^2 + D_2^2 \geq C \left(L_1^2 + L_2^2 \right)$, as in the proof of Lemma 4.2 – estimate (4.6). Hence the dissipation estimate (4.16) becomes

$$\frac{\mathrm{d}\widetilde{L}}{\mathrm{d}t}(t) \leq - C \left(L_1(t)^2 + L_2(t)^2 \right) + h(\mu_1(t), \mu_2(t)) + R(t),$$

where

$$R(t) = -\log \frac{n_1(t,0)}{\mu_1}(n_1(t,0) - \mu_1) - \log \frac{n_2(t,0)}{\mu_2}(n_2(t,0) - \mu_2)$$
$$+ \chi \frac{(\mu_1 - \mu_2)^2}{4}(n_1(t,0) + n_2(t,0)), \quad (4.17)$$

and h is defined by

$$h(\mu_1, \mu_2) = \chi^2 \mu_1 \mu_2 \left(\mu_1 \mu_2 - \mu_+^2 \right) (2 - \mu_1 - \mu_2) - \chi \left(\left(\frac{\mu_1 + \mu_2}{2} \right)^2 - \mu_+^2 \right) (\mu_1 + \mu_2).$$

Since $h(\mu_+, \mu_+) = 0$, $\nabla h(\mu_+, \mu_+) = 0$, and the matrix $\nabla^2 h|_{(\mu_+, \mu_+)}$ is symmetric definite negative, then, locally in the neighborhood of (μ_+, μ_+), there exists a positive constant C such that

$$h(\mu_1, \mu_2) \leq -C \left((\mu_1 - \mu_+)^2 + (\mu_2 - \mu_+)^2 \right) \leq -C \left(f_{\log}(\mu_1(t)) + f_{\log}(\mu_2(t)) \right).$$

Therefore, for $t \geq t_0$ large enough, so that the μ_i are close enough to μ_+, up to a change of the value of the constant $C > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}(t) \leq -C\widetilde{L}(t)^2 + R(t), \quad (4.18)$$

Note that in the symmetric case, it is possible to conclude from this stage, as done previously. However, in the general case, the additional contribution involving $(\mu_1 - \mu_2)^2$ in (4.17) requires more work to handle with. It follows from inequality (4.18) that $\widetilde{L}$ is bounded. Indeed, denoting $F(t) = (1 + t)\widetilde{L}(t)$, we see that the previous inequality (4.18) rewrites as

$$F'(t) \leq \widetilde{L}(t) - C(1 + t)\widetilde{L}(t)^2 + (1 + t)R(t),$$

from which we deduce that

$$F'(t) \leq \frac{1}{1 + t}(F - CF^2) + (1 + t)R(t).$$

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We first notice the obvious upper bound $F - CF^2 \leq (4C)^{-1}$, so that we have

$$F(t) \leq F(0) + \frac{1}{4C}\log(1+t) + \int_0^t (1+s)R(s)\,\mathrm{d}s\,.$$

In order to estimate the last term, we observe that $R = R_1 + R_2$, where

$$R_i(t) := -\log\frac{n_i(t,0)}{\mu_i}(n_i(t,0) - \mu_i) + \chi\frac{(\mu_1 - \mu_2)^2}{4}n_i(t,0)$$

$$= (n_i(t,0) - \mu_i)\left(-\log\frac{n_i(t,0)}{\mu_i} + \chi\frac{(\mu_1 - \mu_2)^2}{4}\right) + \chi\frac{(\mu_1 - \mu_2)^2}{4}\mu_i\,.$$

The first contribution in the right hand-side of the previous equality is nonnegative if and only if

$$\mu_i \leq n_i(t,0) \leq \mu_i e^{\chi\frac{(\mu_1 - \mu_2)^2}{4}}\,.$$

Moreover, we see that

$$R_i(t) \leq \chi\mu_i\frac{(\mu_1 - \mu_2)^2}{4}\left(e^{\chi\frac{(\mu_1 - \mu_2)^2}{4}} - 1\right) + \chi\frac{(\mu_1 - \mu_2)^2}{4}\mu_i\,.$$

Then, using the mean value theorem and adding $R_1 + R_2$, we have

$$R(t) \leq C(\mu_1 - \mu_2)^2 \leq C(\overline{\mu} - \underline{\mu})^2\,,$$

by the comparison principle, where $\overline{\mu}$ and $\underline{\mu}$ are defined as in Section 4.2. Hence recalling (4.15), we deduce that

$$\int_0^t (1+s)R(s)\,\mathrm{d}s \leq C(1 + \log(1+t))\,.$$

So that, we finally obtain

$$F(t) - F(0) \leq C(1 + \log(1+t))\,,$$

and therefore

$$\widetilde{L}(t) \leq C\frac{1 + \log(1+t)}{(1+t)}\,.$$

The statements of Proposition 1.2 follow from this entropy estimate as in Corollary 4.3.

5 Self-similar decay: proof of Proposition 1.3

As in the previous section, we consider successively the symmetric case and the general case.

5.1 Self-similar decay in the symmetric case

It is natural to perform the following parabolic rescaling in order to catch the self-similar behaviour:

$$n(t,x) = \frac{1}{\sqrt{1+2t}}u\left(\frac{1}{2}\log(1+2t), \frac{x}{\sqrt{1+2t}}\right) = \frac{1}{\sqrt{1+2t}}u(\tau, y)\,,$$

which is mass-preserving

$$\int_0^\infty u(\tau, y)\,\mathrm{d}y = \int_0^\infty n\left(\frac{e^{2\tau} - 1}{2}, x\right)\,\mathrm{d}x = 1 - \mu\left(\frac{e^{2\tau} - 1}{2}\right) = 1 - \nu(\tau)\,,$$

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where we have set $\mu(t) = \mu\left(\frac{e^{2\tau}-1}{2}\right) = \nu(\tau)$. Since (n, μ) is solution to (4.1), (u, ν) satisfies the following boundary value problem:

$$\begin{cases} \partial_\tau u(\tau, y) - \partial_{yy} u(\tau, y) - \partial_y(y u(\tau, y)) - \chi\nu(\tau)^2 e^\tau \partial_y u(\tau, y) = 0, & (\tau, y) \in (0, \infty)^2, \\ \frac{d\nu}{d\tau}(\tau) = \partial_y u(\tau, 0) + \chi\nu(\tau)^2 e^\tau u(\tau, 0) = e^\tau u(\tau, 0) - e^{2\tau}\nu(\tau), \end{cases}$$

with the same initial data as in the original variables. Note that the additional left-sided drift $\partial_y(y u(\tau, y))$ contributes to confine the mass in the new frame (τ, y).

Then, Proposition 1.3 is a consequence of the following formulation in self-similar variables (in the symmetric case):

Proposition 5.1. *Under the same assumptions as in Proposition 1.3, the following convergence estimates hold true:*

$$\nu(\tau) + \int_0^\infty u \log \frac{u}{(1-\nu)G} \le Ce^{-\tau}.$$

In particular, it follows from the Csiszar-Kullbáck inequality that

$$\|u - G\|_1 \le Ce^{-\frac{\tau}{2}}.$$

Proof of Proposition 5.1. We split the proof into several intermediate results. We start by establishing bounds on the first and second moments and then we prove entropy dissipation.

Lemma 5.2. *There exist two constants M_1 and M_2 depending only on $\int_0^\infty y^2 u(0, y)\, \mathrm{d}y$ such that*

$$\forall \tau > 0, \quad \int_0^\infty y u(\tau, y)\, \mathrm{d}y \le M_1 \quad \text{and} \quad \int_0^\infty y^2 u(\tau, y)\, \mathrm{d}y \le M_2.$$

Proof of Lemma 5.2. We first see that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^\infty y^2 u(\tau, y)\, \mathrm{d}y = 2(1 - \nu(\tau)) - 2\chi\nu(\tau)^2 e^\tau \int_0^\infty y u(\tau, y)\, \mathrm{d}y - 2\int_0^\infty y^2 u(\tau, y)\, \mathrm{d}y,$$

$$\le 2 - 2\int_0^\infty y^2 u(\tau, y)\, \mathrm{d}y$$

from which it follows that

$$\int_0^\infty y^2 u(\tau, y)\, \mathrm{d}y \le M_2 = \max\left(1, \int_0^\infty y^2 u(0, y)\, \mathrm{d}y\right).$$

Using the Cauchy-Schwarz inequality, we deduce that

$$\int_0^\infty y u(\tau, y)\, \mathrm{d}y \le \sqrt{1 - \nu(\tau)}\sqrt{M_2} \le \sqrt{M_2} =: M_1.$$

$\square$

Let us now prove entropy dissipation. To do so, we compare the solution u to the normalized gaussian G on the half line. We begin with some notations. Let the renormalized density v such that $u(\tau, y) = (1 - \nu(t))v(\tau, y)$. Consider the following Lyapunov functional $\mathcal{L}$:

$$\mathcal{L}(\tau) = \underbrace{(1 - \nu(\tau))\mathcal{H}(v(\tau, \cdot)|G)}_{\mathcal{L}_1} + \underbrace{H(1 - \nu(\tau))}_{\mathcal{L}_2} + \underbrace{G(0)e^{-\tau}H\left(\frac{\nu(\tau)}{G(0)e^{-\tau}}\right)}_{\mathcal{L}_3} + \underbrace{\frac{\chi}{3}\nu(\tau)^3}_{\mathcal{L}_4}, \quad (5.1)$$

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where $H(x) = x \log x - x + 1$ and where we have defined various contributions $\mathcal{L}_i$ that will be studied separately. Since $\mathcal{H}$ and H is non-negative, $\mathcal{L}$ is also non-negative. Notice that $\mathcal{L}$ can be written alternatively as

$$\mathcal{L}(\tau) = \int_0^\infty u(\tau, y) \log \frac{u(\tau, y)}{G(y)} \, \mathrm{d}y + \nu(\tau) \log \frac{\nu(\tau)}{G(0)e^{-\tau}} + G(0)e^{-\tau} + \frac{\chi}{3}\nu(\tau)^3 \,. \qquad (5.2)$$

Finally, let introduce the following auxiliary function that will play a pivotal role:

$$G_{\nu;\tau}(y) = \frac{e^{-\chi\nu^2 e^\tau y - \frac{y^2}{2}}}{\int_0^\infty e^{-\chi\nu^2 e^\tau z - \frac{z^2}{2}} \, \mathrm{d}z} \,. \qquad (5.3)$$

Also, recall the definition of the Fisher information:

$$\mathcal{I}\left(u|G_{\nu(\tau);\tau}\right) = \int_0^\infty u(\tau, y) \left(\partial_y \log u(\tau, y) + y + \chi\nu(\tau)^2 e^\tau\right)^2 \, \mathrm{d}y \geq 0 \,. \qquad (5.4)$$

In the following Lemma we obtain upper bounds on $\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau}(\tau)$.

Lemma 5.3. *The following upper bound holds true:*

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau}(\tau) \leq -\mathcal{I}\left(u|G_{\nu(\tau);\tau}\right) + \nu(\tau) - G(0)e^{-\tau}$$
$$+ \chi\nu(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) \, \mathrm{d}y - \chi\nu(\tau)^3 e^{2\tau} P(\nu(\tau)) \,, \qquad (5.5)$$

where the polynomial function P is defined by (1.7).

Proof of lemma 5.3. Differentiating (5.2), we obtain

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau}(\tau) = \int_0^\infty \log \frac{u(\tau, y)}{G(y)} \partial_\tau u(\tau, y) \, \mathrm{d}y + \left(\log \frac{\nu(\tau)}{G(0)e^{-\tau}} + \chi\nu(\tau)^2\right) \frac{\mathrm{d}\nu}{\mathrm{d}\tau}(\tau) + \nu(\tau) - G(0)e^{-\tau}$$

$$= -\left(\log \frac{u(\tau, 0)}{G(0)}\right) \frac{\mathrm{d}\nu}{\mathrm{d}\tau}(\tau) + \left(\log \frac{\nu(\tau)}{G(0)e^{-\tau}} + \chi\nu(\tau)^2\right) \frac{\mathrm{d}\nu}{\mathrm{d}\tau}(\tau) + \nu(\tau) - G(0)e^{-\tau}$$

$$- \int_0^\infty u(\tau, y) \left(\partial_y \log u(\tau, y) + y\right) \left(\partial_y \log u(\tau, y) + y + \chi\nu(\tau)^2 e^\tau\right) \, \mathrm{d}y$$

$$= -\left(\log \frac{u(\tau, 0)}{\nu(\tau)e^\tau}\right) \frac{\mathrm{d}\nu}{\mathrm{d}\tau}(\tau) + \nu(\tau) - G(0)e^{-\tau} + \chi\nu(\tau)^2 e^\tau u(\tau, 0) - \chi\nu(\tau)^3 e^{2\tau}$$

$$- \int_0^\infty u(\tau, y) \left(\partial_y \log u(\tau, y) + y\right) \left(\partial_y \log u(\tau, y) + y + \chi\nu(\tau)^2 e^\tau\right) \, \mathrm{d}y \,.$$

Using the definition (5.4) of $\mathcal{I}\left(u|G_{\nu(\tau);\tau}\right)$, we have

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau}(\tau) \leq -\mathcal{I}\left(u|G_{\nu(\tau);\tau}\right) + \nu(\tau) - G(0)e^{-\tau} + \chi\nu(\tau)^2 e^\tau u(\tau, 0) - \chi\nu(\tau)^3 e^{2\tau}$$

$$+ \chi\nu(\tau)^2 e^\tau \int_0^\infty u(\tau, y) \left(\partial_y \log u(\tau, y) + y + \chi\nu(\tau)^2 e^\tau\right) \, \mathrm{d}y$$

$$\leq -\mathcal{I}\left(u|G_{\nu(\tau);\tau}\right) + \nu(\tau) - G(0)e^{-\tau}$$

$$+ \chi\nu(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) \, \mathrm{d}y + \left(\chi\nu(\tau)^2 e^\tau\right)^2 (1 - \nu(\tau)) - \chi\nu(\tau)^3 e^{2\tau}$$

$$\leq -\mathcal{I}\left(u|G_{\nu(\tau);\tau}\right) + \nu(\tau) - G(0)e^{-\tau}$$

$$+ \chi\nu(\tau)^2 e^\tau \int_0^\infty yu(\tau, y) \, \mathrm{d}y - \chi\nu(\tau)^3 e^{2\tau} P(\nu(\tau)) \,,$$

which is the estimate (5.5). □

Several lower bound on the relative entropy are required to control entropy dissipation.

Lemma 5.4. *The following inequality holds true*

$$\mathcal{I}\left(u|G_{\nu;\tau}\right) \geq 2\mathcal{L}_1 - 2\chi\nu^2 e^\tau \int_0^\infty y\left((1-\nu)G - u\right)\,\mathrm{d}y\,.$$

Proof of Lemma 5.4. Recalling the definition (5.3) of $G_{\nu;\tau}$, the logarithmic Sobolev inequality (Lemma B.5) yields that

$$\mathcal{I}\left(u|G_{\nu;\tau}\right) \geq 2\int_0^\infty u\log\frac{u}{(1-\nu)G_{\nu;\tau}}\,\mathrm{d}y\,.$$

The latter term can be decomposed as follows,

$$\int_0^\infty u\log\frac{u}{(1-\nu)G_{\nu;\tau}}\,\mathrm{d}y = \underbrace{\int_0^\infty u\log\frac{u}{(1-\nu)G}\,\mathrm{d}y}_{\mathcal{L}_1} + \int_0^\infty u\log\frac{G}{G_{\nu;\tau}}\,\mathrm{d}y\,.$$

Using Jensen's inequality, we have

$$\log\frac{G}{G_{\nu;\tau}} = \chi\nu^2 e^\tau y + \log\frac{\int_0^\infty e^{-\chi\nu^2 e^\tau z - \frac{z^2}{2}}\,\mathrm{d}z}{\int_0^\infty e^{-\frac{z^2}{2}}\,\mathrm{d}z}$$

$$\geq \chi\nu^2 e^\tau y - \int_0^\infty \chi\nu^2 e^\tau z G(z)\,\mathrm{d}z\,.$$

This completes the proof of Lemma 5.4. □

We can now derive a quantitative rate of convergence.

Lemma 5.5. *There exists a positive constant C such that*

$$\mathcal{L}(\tau) \leq Ce^{-\tau}\,.$$

Proof of Lemma 5.5. Injecting the result of Lemma 5.4 into inequality (5.5), it follows that

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau}(\tau) \leq -2\mathcal{L}_1 + \nu - G(0)e^{-\tau} - \chi\nu^2 e^\tau \int_0^\infty yu\,\mathrm{d}y$$

$$+ 2\chi\nu^2 e^\tau \int_0^\infty y(1-\nu)G\,\mathrm{d}y - \chi\nu^3 e^{2\tau}P(\nu)$$

$$\leq -2\mathcal{L}_1(\tau) + \nu - G(0)e^{-\tau} + 2\chi\nu^2 e^\tau(1-\nu)G(0) - \chi\nu^3 e^{2\tau}P(\nu)\,.$$

Under the assumptions of Proposition 5.1, there exists $P_0 > 0$ such that $P(\nu(\tau)) \geq P_0$ for all $\tau \geq 0$. Indeed, either $\chi \geq 4$ and $P(\mu(t)) \geq P(\mu_0)$ by the comparison principle (Lemma 3.2), or $\chi < 4$ and P is uniformly bounded from below.

To conclude, let add $2\mathcal{L}$ on both sides in order to get

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tau}(\tau) + 2\mathcal{L}(\tau) \leq \nu - G(0)e^{-\tau} - \chi\nu^3 e^{2\tau}P_0 + 2\chi G(0)\nu^2 e^\tau + 2\mathcal{L}_2(\tau) + 2\mathcal{L}_3(\tau) + 2\mathcal{L}_4(\tau)\,.$$

From the definitions of $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ (5.1), we see that

$$\mathcal{L}_2(\tau)e^{2\tau} = ((1-\nu)\log(1-\nu)+\nu)\,e^{2\tau} \leq \nu e^{2\tau}\,,$$

$$\mathcal{L}_3(\tau)e^{2\tau} = G(0)e^{\tau}H\left(\frac{\nu e^{\tau}}{G(0)}\right) = \nu e^{2\tau}\log\frac{\nu e^{\tau}}{G(0)} - \nu e^{2\tau} + G(0)e^{\tau}$$

$$\mathcal{L}_4(\tau)e^{2\tau} = \frac{\chi}{3}\nu^3 e^{2\tau} \leq \frac{\chi}{3}\nu e^{2\tau}\,.$$

Consequently, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(e^{2\tau}\mathcal{L}(\tau)) \leq e^{\tau}h(\nu e^{\tau}),$$

where the function h is defined by

$$h(z) = -\chi P_0 z^3 + 2\chi G(0)z^2 + 2z\log\frac{z}{G(0)} + G(0) + z + \frac{\chi}{3}z\,.$$

The function h is clearly bounded from above. Therefore, after integration between 0 and τ, we obtain

$$\mathcal{L}(\tau)e^{2\tau} \leq \mathcal{L}(0) + e^{\tau}\sup h\,,$$

$$\mathcal{L}(\tau) \leq (\mathcal{L}(0) + \sup h)\,e^{-\tau}\,.$$

This concludes the proofs of Lemma 5.5 and Proposition 5.1. $\qquad\qquad\qquad\square$

5.2 Self-similar decay in the nonsymmetric case

In the general case, the solution can be bounded from above by the special solution of the symmetric problem initialized by (N_{μ_0}, N_{μ_0}), due to the comparison principle (Lemma 3.2 and 3.3). We deduce the following rate of convergence:

$$\nu_i(\tau) \leq Ce^{-\tau}\,, \quad i = 1, 2\,.$$

With similar notations as before, we define the Lyapunov functionals $\mathcal{L}_{\mathrm{gen}}^{(i)}$ by

$$\mathcal{L}_{\mathrm{gen}}^{(i)}(\tau) = \mathcal{L}_1^{(i)} + \mathcal{L}_2^{(i)} + \mathcal{L}_3^{(i)}\,, \quad i = 1, 2\,,$$

where each $\mathcal{L}_j^{(i)}$ is defined as in (5.1), but involving u_i, ν_i. An important point here is that it does not contain the correction term $\mathcal{L}_4$. Following the previous computations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{L}_{\mathrm{gen}}^{(i)}(\tau) \leq -\mathcal{I}^{(i)} + \nu_i - G(0)e^{-\tau} - \chi\nu_1\nu_2 e^{\tau}u_i(\tau,0) + \chi\nu_i\nu_1\nu_2 e^{2\tau}$$

$$+ \chi\nu_i^2 e^{\tau}\int_0^{\infty} yu_i(\tau,y)\,\mathrm{d}y - \chi\nu_i\nu_1\nu_2 e^{2\tau}P(\nu_i(\tau))\,,$$

where we have denoted

$$\mathcal{I}^{(i)} = \int_0^{\infty} u_i\left(\partial_y\log y + y + \chi\nu_1\nu_2 e^{\tau}\right)^2\,\mathrm{d}y\,.$$

Using the estimate $\nu_i \leq Ce^{-\tau}$, this leads to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{L}_{\mathrm{gen}}^{(i)}(\tau) \leq -\mathcal{I}^{(i)} + Ce^{-\tau}\,.$$

Using similar computations as before, we obtain

$$\mathcal{L}_{\mathrm{gen}}^{(i)}(\tau) \leq Ce^{-\tau}\,.$$

This concludes the proof of Proposition 1.3.

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A Linear stability analysis: proof of Proposition 1.1

The linear stability analysis involves a series of Lemma.

Lemma A.1. *The steady states $(\overline{n}_i, \overline{\mu}_i)_{i=1,2}$ of the system (1.2)–(1.3)–(1.4) with (1.5) satisfy $\overline{n}_i = \overline{n}$ and $\overline{\mu}_i = \overline{\mu}$, for $i = 1, 2$, where $(\overline{n}, \overline{\mu})$ is solution to*

$$\begin{cases} \overline{n}(x) = \overline{\mu} \exp\left(-\chi\,\overline{\mu}^2\,x\right), & x \geq 0, \\ \quad 0 = \chi\,\overline{\mu}^2 - \chi\,\overline{\mu} + 1. \end{cases} \tag{A.1}$$

Such a steady state exists iff $\chi \geq 4$. Moreover, in the case where $\chi > 4$, there are two steady states, $(\mu_- e^{-\chi\mu_-^2 x}, \mu_-)$ and $(\mu_+ e^{-\chi\mu_+^2 x}, \mu_+)$ with $0 < \mu_- < \mu_+ < 1$.

Proof. The only part to prove is that any steady state has identical values for the two cells. A straightforward computation yields that $(\overline{n}_i, \overline{\mu}_i)_{i=1,2}$ satisfies, for $i = 1$ and 2, the following system:

$$\begin{cases} \overline{n}_i(x) = \overline{\mu}_i \exp\left(-\chi\,\overline{\mu}_1\overline{\mu}_2\,x\right), & x \geq 0, \\ \quad 0 = \chi\,\overline{\mu}_1\overline{\mu}_2 - \chi\,\overline{\mu}_i + 1. \end{cases}$$

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From the second equation in the previous system it follows that $\overline{\mu}_1 = \overline{\mu}_2$, hence it yields to (A.1) which admits a real solution $\overline{\mu}$ iff $\chi \geq 4$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma A.2. *Assume that $\chi > 4$, and let $\overline{\mu} \in \{\mu_-, \mu_+\}$. The linearized system associated to (1.2)–(1.3)–(1.4) with (1.5) does not admit 0 as an eigenvalue. Moreover, a complex $\lambda \neq 0$ satisfying $\Re(\lambda) \geq 0$ is an eigenvalue of the linear system if and only if it satisifies*

$$\left(\chi\overline{\mu}^2 + \beta_1\right)\left(\lambda + 2\,\frac{\chi^2\,\overline{\mu}^4}{\lambda} + 1\right) + 2\,\chi\,\overline{\mu}^2 - \lambda = 0\,, \tag{A.2}$$

where β_1 is the unique root of the equation $X^2 + \chi\overline{\mu}^2 X - \lambda = 0$ satisfying $\Re(\beta_1) < -\chi\overline{\mu}^2$.

Proof of Lemma A.2. Let us consider zero mass perturbations around the steady state $(\overline{n}, \overline{\mu})$ solution to (A.1). For $i = 1$ or 2, we seek the first order modes as follows

$$\begin{cases} n_i(t, x) = \overline{n}(x) + \widetilde{n}_i(x)\exp\left(\lambda\,t\right) + \textit{higher order terms}\,, \quad x \geq 0\,, \\ \mu_i(t) = \overline{\mu} + \widetilde{\mu}_i\exp\left(\lambda\,t\right) + \textit{higher order terms}\,, \\ 0 = \displaystyle\int_0^{+\infty} \widetilde{n}_i(t, x)\,\mathrm{d}x + \widetilde{\mu}_i\,, \end{cases} \tag{A.3}$$

where $\lambda \in \mathbb{C}$ is some eigenvalue. We linearize (1.2)–(1.3)–(1.4) and we obtain two systems for $i = 1, 2$:

$$\begin{cases} \lambda\widetilde{n}_i(x) = \widetilde{n}_i''(x) + \chi\overline{\mu}^2\,\widetilde{n}_i'(x) - \chi^2\,\overline{\mu}^4\exp\left(-\chi\,\overline{\mu}^2\,x\right)(\widetilde{\mu}_1 + \widetilde{\mu}_2)\,, \\ \lambda\widetilde{\mu}_i = \widetilde{n}_i(0) - \widetilde{\mu}_i\,, \\ \lambda\widetilde{\mu}_i = \widetilde{n}_i'(0) + \chi\overline{\mu}^2\,(\widetilde{n}_i(0) + \widetilde{\mu}_1 + \widetilde{\mu}_2)\,. \end{cases} \tag{A.4}$$

We first investigate the condition for having $\Re(\lambda) \geq 0$ (linear instability). In the case where $\chi > 4$, the roots of $X^2 + \chi\,\overline{\mu}^2\,X - \lambda = 0$ are

$$\beta_1 = \frac{-\chi\,\overline{\mu}^2 - \sqrt{\chi^2\,\overline{\mu}^4 + 4\lambda}}{2} \quad \text{and} \quad \beta_2 = \frac{-\chi\,\overline{\mu}^2 + \sqrt{\chi^2\,\overline{\mu}^4 + 4\lambda}}{2}\,.$$

Here we have abusively denoted by $\sqrt{\chi^2\,\overline{\mu}^4 + 4\lambda}$ the only complex number with positive real part satisfying $z^2 = \chi^2\,\overline{\mu}^4 + 4\lambda$. With these notations, we have $\Re(\beta_1) \leq -\chi\overline{\mu}^2$, $\Re(\beta_2) \geq 0$, the inequalities being strict as soon as we have $\lambda \neq 0$. Hence, the solution to (A.4) can be written as:

$$\begin{cases} \widetilde{n}_i(x) = C_i\,\exp(\beta_1\,x) + D_i\,\exp(\beta_2\,x) - \frac{\chi^2\,\overline{\mu}^4}{\lambda}\exp\left(-\chi\,\overline{\mu}^2\,x\right)(\widetilde{\mu}_1 + \widetilde{\mu}_2)\,, & \text{if } \lambda \neq 0 \\ \widetilde{n}_i(x) = C_i\,\exp(-\chi\,\overline{\mu}^2\,x) + D_i - \chi\,\overline{\mu}^2\,x\exp\left(-\chi\,\overline{\mu}^2\,x\right)(\widetilde{\mu}_1 + \widetilde{\mu}_2)\,, & \text{if } \lambda = 0 \end{cases}$$

The perturbations must be integrable, so that $D_i = 0$.

We first establish that $\lambda = 0$ cannot be an eigenvalue. Indeed from (A.4) we deduce that $C_i = \widetilde{\mu}_i$. Then, the zero integrability condition in (A.3) yields that $\widetilde{\mu}_i = \chi\overline{\mu}^2\widetilde{\mu}_j$, $i \neq j$, and thus $\chi^2\overline{\mu}^4 = 1$ if $\widetilde{\mu}_i \neq 0$. This can only happen if $\chi = 4$ and $\overline{\mu} = \frac{1}{2}$. Consequently 0 is not an eigenvalue for the linearized system for $\chi > 4$. From now on, we can assume that

$$\Re(\lambda) \geq 0, \quad \lambda \neq 0, \quad \Re(\beta_1) < -\chi\overline{\mu}^2\,.$$

The last two equations of (A.4) now read as the two systems

$$\begin{cases} (\lambda + 1)\,\widetilde{\mu}_i - C_i = -\frac{\chi^2\,\overline{\mu}^4}{\lambda}(\widetilde{\mu}_1 + \widetilde{\mu}_2)\,, \\ \lambda\,\widetilde{\mu}_i - \left(\chi\,\overline{\mu}^2 + \beta_1\right)C_i = \chi\,\overline{\mu}^2\,(\widetilde{\mu}_1 + \widetilde{\mu}_2)\,. \end{cases} \quad,\quad i = 1, 2\,. \tag{A.5}$$

Since $\chi\overline{\mu}^2\beta_1 + \beta_1^2 = \lambda$, we notice that

$$\det\begin{pmatrix} \lambda+1 & -1 \\ \lambda & -(\chi\overline{\mu}^2+\beta_1) \end{pmatrix} = \underbrace{(\chi\overline{\mu}^2+\beta_1)}_{\neq 0}\underbrace{(-\lambda-1+\beta_1)}_{\Re(\cdot)<-\chi\overline{\mu}^2-1} \neq 0\,.$$

This provides the additional informations $C_1 = C_2 = C$ and $\widetilde{\mu}_1 = \widetilde{\mu}_2 = \widetilde{\mu}$. Writing back system (A.5) in terms of C, μ only, the existence of a nonzero solution to (A.5) is then equivalent to the degeneracy of the obtained system i.e. (A.2) is fullfilled. $\qquad\square$

Lemma A.3. *There exists an eigenvalue with nonnegative real part $\Re(\lambda) \geq 0$ if and only if $\chi\overline{\mu}^2 \leq 1$.*

Proof of lemma A.3. We argue by contradiction. Denoting by $x = \frac{\lambda}{\chi^2\,\mu^4}$ and $\gamma = \chi\,\overline{\mu}^2$ and recalling the expression of β_1, equation (A.2) is equivalent to the following one:

$$\frac{\gamma}{2}\left(\left(1-\sqrt{1+4\,x}\right)\left(\gamma^2\,x + \frac{2}{x} + 1\right) + 4 - 2\,\gamma\,x\right) = 0\,. \tag{A.6}$$

It is convenient to introduce $2z = \sqrt{1+4x} - 1$, such that $4x = (2z+1)^2 - 1 = 4z(z+1)$. Notice that λ has a nonnegative real part $\Re(\lambda) \geq 0$ if and only if z has a nonnegative real part $\Re(z) \geq 0$. The equation (A.6) is equivalent to the following series of equations, since $z \neq 0$:

$$-2z\left(\gamma^2 z(z+1) + \frac{2}{z(z+1)} + 1\right) + 4 - 2\gamma z(z+1) = 0$$

$$-2\gamma^2 z^2(z+1) + \frac{4z}{(z+1)} - 2z - 2\gamma z(z+1) = 0$$

$$-\gamma^2 z(z+1)^2 + 2 - (z+1) - \gamma(z+1)^2 = 0$$

$$\gamma^2 z^3 + \gamma(2\gamma+1)z^2 + (\gamma+1)^2 z + \gamma - 1 = 0\,.$$

By monotonicity, the latter cubic polynomial has a nonnegative real root if and only if $\gamma \leq 1$, *i.e.* $\chi\overline{\mu}^2 \leq 1$. In the case $\gamma > 1$, it cannot have a complex root ω such that $\Re(\omega) \geq 0$ either. Suppose it is the case. Then, $\overline{\omega}$ is another root, and the relations between roots and coeffients imply that

$$-\gamma^2(2\Re(\omega)+\xi) = \gamma(2\gamma+1) \quad\Longleftrightarrow\quad 2\Re(\omega) = -\xi - 2 - \frac{1}{\gamma}\,,$$

where ξ is the real root of the cubic polynomial. To conclude, it is enough to show that $\xi > -2$. Indeed, by evaluating the cubic polynomial at $z = -2$, we obtain:

$$-8\gamma^2 + 4\gamma(2\gamma+1) - 2(\gamma+1)^2 + \gamma - 1 = -2\gamma^2 + \gamma - 3 < 0\,.$$

This concludes the proof of Lemma A.3. $\qquad\square$

The stability results stated in Proposition 1.1 follows from Lemma A.1 and Lemma A.3, and the following observation about the roots of P:

$$\gamma_- = \chi\mu_-^2 < 1 < \gamma_+ = \chi\mu_+^2\,.$$

B Definitions and useful inequalities

Let us recall some classical definitions.

Definition B.1. *Given two probability measures p, q on $\mathbb{R}_+$, we define the relative entropy of p with respect to q by*

$$\mathcal{H}(p|q) = \int_0^\infty p(x) \log \frac{p(x)}{q(x)} \, \mathrm{d}x = \int_0^\infty q(x) \left(\frac{p(x)}{q(x)} \log \frac{p(x)}{q(x)} - \frac{p(x)}{q(x)} + 1 \right) \mathrm{d}x \geq 0 \,.$$

The Fisher information of p with respect to q is defined as the quantity

$$\mathcal{I}(p|q) = \int_0^\infty p(x) \left(\partial_x \log \frac{p(x)}{q(x)} \right)^2 \mathrm{d}x \,.$$

Moreover if p, q have finite second moment, [37], the quadratic Wasserstein distance $W(p,q)$ is defined by

$$W(p,q) = \sqrt{\inf_{\pi \in \Pi(p,q)} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |x - y|^2 \, \mathrm{d}\pi(x,y)} \,,$$

where $\Pi(p,q)$ denotes the set of probability measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with marginals p and q.

Finally, we consider similarly another Wasserstein distance

$$W_1(p,q) = \inf_{\pi \in \Pi(p,q)} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |x - y| \, \mathrm{d}\pi(x,y) \,,$$

There are several results concerning possible representations of Wasserstein distances when it is specialized to the real line. In such a case it can be considerably simplified in terms of the distribution functions $F(x) = \int_0^x p(y) \, \mathrm{d}y$, $x \in (0, \infty)$, associated to probability measures p, [37].

Theorem B.2 (Representation for W)**.** *Let p and q be probability measures on $\mathbb{R}_+$ with respective distribution functions F and G. Then*

$$W^2(p,q) = \int_0^1 |F^{-1}(t) - G^{-1}(t)|^2 \, \mathrm{d}t \,,$$

and

$$W_1(p,q) = \int_0^1 |F^{-1}(t) - G^{-1}(t)| \, \mathrm{d}t = \int_0^\infty |F(x) - G(x)| \, \mathrm{d}x \,,$$

where F^{-1} is the pseudo-inverse function:

$$F^{-1}(t) = \inf\{x \in \mathbb{R} \,:\, F(x) \geq t\} \,, \ 0 < t < 1.$$

Lemma B.3. *Let $(\overline{p}, \underline{p})$, and (n, m) be two pairs of probability measures on $\mathbb{R}_+$ with distribution functions $(\overline{P}, \underline{P})$ and (N, M), respectively. Suppose that they are ordered such that $\underline{P} \leq N, M \leq \overline{P}$, then the Wasserstein distances are ordered as follows,*

$$W(n, m) \leq W\left(\underline{p}, \overline{p}\right) \,, \quad W_1(n, m) \leq W_1\left(\underline{p}, \overline{p}\right) \,.$$

The following lemma can be applied to interpolation between p and δ_0.

Lemma B.4. *Let p, q be two probability measures on $\mathbb{R}_+$ with finite r moment for $r \in \{1, 2\}$ and let $0 < \mu < 1$, then*

$$W_r(\mu\delta_0 + (1 - \mu)p, \mu\delta_0 + (1 - \mu)q) = (1 - \mu)^{1/r}W(p, q).$$

Proof. This results is of course valid for any $r \geq 1$. It can be established by direct computation on the pseudo inverse. $\square$

The two following inequalities are useful to link the relative entropy, and the Fisher information arising as a part of the entropy dissipation, see [37].

Lemma B.5 (Log-Sobolev inequality). *Assume that q satisfies a Gaussian concentration principle i.e. $q(x) = e^{-V(x)}$ with $V''(x) \geq \alpha > 0$, then the logarithmic Sobolev inequality holds true*

$$\mathcal{I}(p|q) \geq 2\alpha\mathcal{H}(p|q).$$

Lemma B.6 (HWI inequality for exponential measure). *Assume that $q(x) = \lambda e^{-\lambda x}$ then the following inequality holds true*

$$\mathcal{I}(p|q) \geq \left(\frac{\mathcal{H}(p|q)}{W(p, q)}\right)^2.$$

Usually, convergence in the L^1 norm is controlleb by the entropy by means of the Csiszár-Kullback inequality, [15, 25].

Proposition B.7 (Csiszár-Kullback inequality). *For any non-negative functions $f, g \in L^1(\mathbb{R}_+)$ such that $\int_{\mathbb{R}_+} f(x)\,\mathrm{d}x = \int_{\mathbb{R}_+} g(x)\,\mathrm{d}x = M$, we have that*

$$\|f - g\|_1^2 \leq 2M \int_0^\infty f(x) \log\left(\frac{f(x)}{g(x)}\right)\,\mathrm{d}x.$$

C Well-posedness of the coupled PDE-ODE system

Well-posedness of the Cauchy problem (1.2)–(1.3)–(1.4) relies on a fixed point theorem, as usual. For the sake of clarity, we restrict to a single coupling between the cell bulk density $n(t, x)$ and the boundary value $\mu(t)$. It contains the principal technical difficulty which arises from the coupling between the ODE for μ and the PDE for n. We believe that the extension to the case of pairwise interactions between two cells is straightforward based on the estimates derived below.

We proceed into two steps. Firstly, we obtain refined entropy estimates for the sole PDE problem, without the coupling. As such, we consider two given functions $\mu_1(t)$ and $\mu_2(t)$, and we derive suitable contraction estimates on n_1, n_2. We introduce the coupling in a second step.

C.1 Entropy estimates for the uncoupled PDE

Given the function μ, and a smooth nonnegative function χ, we consider the solution of the problem

$$\begin{cases} \partial_t n(t, x) - \partial_{xx} n(t, x) - \chi(\mu(t))\partial_x n(t, x) = 0, \\ \partial_x n(t, 0) + \chi(\mu(t))n(t, 0) = n(t, 0) - \mu(t), \\ n(0, x) = n^0(x). \end{cases} \tag{C.1}$$

It is classical that if we consider $\mu \geq 0$ and nonnegative initial data n^0, then there exists a unique solution which is also nonnegative. Moreover the following estimate

$$\frac{d}{dt} \int (1+x) n \, dx = \mu(t) - \chi(\mu(t)) \int n \, dx,$$

ensures that $\int (1+x) n \, dx \leq \int (1+x) n^0 \, dx + \int_0^t \mu \, dt$. Also, the classical entropy estimate

$$\frac{d}{dt} \int (n \log n - n) \, dx = \tag{C.2}$$

In order to compare two solutions n_1, n_2 associated with two inputs μ_1, μ_2, we introduce the following Gajewski metric [20]:

$$d_G(n_1, n_2) = \int \underbrace{H(n_1) + H(n_2) - 2H \left(\frac{n_1 + n_2}{2} \right)}_{\Delta_H(n_1, n_2) \geq 0} dx,$$

where H is the convex function $H(a) = a \log a - a + 1$. In particular, for all a, b, we have:

$$H(b) - H \left(\frac{a+b}{2} \right) = H' \left(\frac{a+b}{2} \right) \left(\frac{b-a}{2} \right) + \frac{1}{2} H''(c) \left(\frac{b-a}{2} \right)^2, \tag{C.3}$$

for some $c \in [a, b]$, and a similar estimate at point a. This yields

$$\Delta_H(a, b) = H(a) + H(b) - 2H \left(\frac{a+b}{2} \right) \geq \frac{1}{4} \frac{(b-a)^2}{\max(a, b)} \geq \frac{1}{4} \frac{(b-a)^2}{a+b}, \tag{C.4}$$

and similarly

$$\Delta_H(a, b) \leq \frac{1}{4} \frac{(b-a)^2}{\min(a, b)}. \tag{C.5}$$

This pointwise inequality turns into the following integral inequality by Cauchy-Scwhwarz:

$$d_G(n_1, n_2) \geq \frac{1}{4} \int \frac{(n_1 - n_2)^2}{n_1 + n_2} \, dx \geq \frac{1}{4} \left(\int n_1 \, dx + \int n_2 \, dx \right)^{-1} \left(\int n_1 \, dx - \int n_2 \, dx \right)^2. \tag{C.6}$$

On the other hand, we have

$$\frac{d}{dt} (d_G(n_1, n_2)) = \int \log \left(\frac{2n_1}{n_1 + n_2} \right) \partial_t n_1 + \log \left(\frac{2n_2}{n_1 + n_2} \right) \partial_t n_2$$

$$= -\log \left(\frac{2n_1(0)}{n_1(0) + n_2(0)} \right) (n_1(0) - \mu_1) - \log \left(\frac{2n_2(0)}{n_1(0) + n_2(0)} \right) (n_2(0) - \mu_2)$$

$$- \int \partial_x \log \left(\frac{2n_1}{n_1 + n_2} \right) (\partial_x n_1 + \chi_1 n_1) - \int \partial_x \log \left(\frac{2n_2}{n_1 + n_2} \right) (\partial_x n_2 + \chi_2 n_2)$$

In the sequel, we use the notation $q_i = \frac{2n_i}{n_1 + n_2}$ and notice that $q_1 + q_2 = 2$ and thus $\partial_x q_1 = -\partial_x q_2$. The previous estimate is equivalent to the following one:

$$\frac{d}{dt} d_G(n_1, n_2) = \mu_1 \log q_1(0) + \mu_2 \log q_2(0) - \Delta_H(n_1(0), n_2(0))$$

$$- \int \left(\frac{n_1 + n_2}{2} \right) \partial_x q_1 (\partial_x \log n_1 + \chi_1) - \int \left(\frac{n_1 + n_2}{2} \right) \partial_x q_2 (\partial_x \log n_2 + \chi_2).$$

To bound the cross term involving μ_i and $n_i(0)$ in the first line, we notice that $\mu_1 \log q + \mu_2 \log(2-q)$ reaches a maximum for the critical value q^* satisfying

$$\frac{\mu_1}{q^*} = \frac{\mu_2}{2 - q^*}, \quad q^* = \frac{2\mu_1}{\mu_1 + \mu_2}.$$

Therefore, the first line is upper bounded by

$$\Delta_H(\mu_1, \mu_2) = \mu_1 \log\left(\frac{2\mu_1}{\mu_1 + \mu_2}\right) + \mu_2 \log\left(\frac{2\mu_2}{\mu_1 + \mu_2}\right)$$

We arrive at

$$\frac{d}{dt} d_G(n_1, n_2) \leq \Delta_H(\mu_1, \mu_2) - \Delta_H(n_1(0), n_2(0))$$
$$- \int \left(\frac{n_1 + n_2}{2}\right) \partial_x q_1 (\partial_x \log n_1 + \chi_1) - \int \left(\frac{n_1 + n_2}{2}\right) \partial_x q_2 (\partial_x \log n_2 + \chi_2)$$
$$= \Delta_H(\mu_1, \mu_2) - \Delta_H(n_1(0), n_2(0))$$
$$- \int \left(\frac{n_1 + n_2}{2}\right) \partial_x q_1 \left(\partial_x \log\left(\frac{n_1}{n_2}\right) + \chi_1 - \chi_2\right).$$

We recall that $\frac{n_1}{n_2} = \frac{q_1}{2 - q_1}$, so that $\partial_x \log\frac{n_1}{n_2} = \partial_x q_1 \left(\frac{1}{q_1} + \frac{1}{2 - q_1}\right)$. Hence we obtain

$$\frac{d}{dt} d_G(n_1, n_2) \leq \Delta_H(\mu_1, \mu_2) - \Delta_H(n_1(0), n_2(0))$$
$$- \int \left(\frac{n_1 + n_2}{2}\right) \left(\frac{1}{q_1} + \frac{1}{2 - q_1}\right) |\partial_x q_1|^2 - (\chi_1 - \chi_2) \int \left(\frac{n_1 + n_2}{2}\right) \partial_x q_1$$

Using again the relationship $q_1 + q_2 = 2$, we can rewrite the last but one integral term as

$$\int \left(\frac{n_1 + n_2}{2}\right) \left(\frac{1}{q_1} + \frac{1}{2 - q_1}\right) |\partial_x q_1|^2 = \int \left(\frac{n_1 + n_2}{2}\right) \left(q_1 |\partial_x \log q_1|^2 + q_2 |\partial_x \log q_2|^2\right)$$
$$= \int n_1 |\partial_x \log q_1|^2 + \int n_2 |\partial_x \log q_2|^2.$$

In addition, we notice that

$$\left|(\chi_1 - \chi_2) \int \left(\frac{n_1 + n_2}{2}\right) \partial_x q_1\right| \leq |\chi_1 - \chi_2| \int \left(\frac{n_1 + n_2}{2}\right) |q_1| |\partial_x \log q_1|$$
$$\leq \frac{|\chi_1 - \chi_2|^2}{2} \int n_1 + \frac{1}{2} \int n_1 |\partial_x \log q_1|^2$$

Combining the integral terms, and symmetrizing the roles of n_1, n_2, we end up with our key estimate:

$$\frac{d}{dt} d_G(n_1, n_2) \leq \Delta_H(\mu_1, \mu_2) - \Delta_H(n_1(0), n_2(0)) + \frac{|\chi_1 - \chi_2|^2}{2} \int \frac{n_1 + n_2}{2}$$
$$- \frac{3}{4} \left(\int n_1 |\partial_x \log q_1|^2 + \int n_2 |\partial_x \log q_2|^2\right) \quad \text{(C.7)}$$

C.2 The fixed point mapping

For a given function $\mu(t)$, we define $n(t,x)$ as in Section C.1. Next, we define $\mu^{\text{out}}(t)$ as a solution of the following ODE:

$$\frac{d}{dt}\mu^{\text{out}}(t) = n(t,0) - \mu^{\text{out}}(t), \quad \text{or equivalently} \quad \mu^{\text{out}}(t) = e^{-t}\mu^{\text{out}}(0) + \int_0^t e^{s-t}n(s,0)\,ds \tag{C.8}$$

with the initial data $\mu^{\text{out}}(0) = 1 - \int n^0(x)\,dx > 0$. On the other hand, $n(t,x)$ and $\mu(t)$ satisfy the following relationship, obtained by integrating (C.1) with respect to space:

$$\frac{d}{dt}\int n(t,x)\,dx = \mu(t) - n(t,0). \tag{C.9}$$

Consequently, (C.8) can be recast as:

$$\mu^{\text{out}}(t) = e^{-t}\mu^{\text{out}}(0) + \int_0^t e^{s-t}\mu(s)\,ds + \int_0^t e^{s-t}I(s)\,ds - I(t) + e^{-t}I(0), \quad I(t) = \int n(t,x)\,dx. \tag{C.10}$$

Suppose we are given two input functions μ_i, $i = 1,2$, then the two output functions μ_i^{out}, $i = 1,2$ share the same initial data. Moreover, denoting $I_i = \int n_i(t,x)\,dx$, we have

$$\mu_1^{\text{out}}(t) - \mu_2^{\text{out}}(t) = I_2(t) - I_1(t) + \int_0^t e^{s-t}(I_1(s) - I_2(s))ds + \int_0^t e^{s-t}(\mu_1(s) - \mu_2(s))ds \tag{C.11}$$

We deduce from our key estimate (C.7) that

$$d_G(n_1(t), n_2(t)) \leq \int_0^t \Delta_H(\mu_1(s), \mu_2(s))\,ds + \frac{1}{4}(\text{Lip}\chi)^2 \int_0^t |\mu_1(s) - \mu_2(s)|^2(I_1(s) + I_2(s))\,ds, \tag{C.12}$$

where $\text{Lip}\chi = \sup_{\mu \in (0,1)} |\chi'(\mu)|$. Finally, using the two pointwise inequalities (C.3)–(C.4), we obtain

$$\frac{(I_1(t) - I_2(t))^2}{\max(I_1(t), I_2(t))} \leq \int_0^t \frac{(\mu_1(s) - \mu_2(s))^2}{\min(\mu_1(s), \mu_2(s))}\,ds + (\text{Lip}\chi)^2 \int_0^t |\mu_1(s) - \mu_2(s)|^2(I_1(s) + I_2(s))\,ds. \tag{C.13}$$

By the combination of (C.11) and (C.13), we obtain a contraction estimate of the form

$$\sup_{t \in [0,T]} |\mu_1^{\text{out}}(t) - \mu_2^{\text{out}}(t)| \leq \omega(T) \sup_{t \in (0,T)} |\mu_1(t) - \mu_2(t)|, \tag{C.14}$$

where $\omega(T) = \mathcal{O}(\sqrt{T})$, provided that $\mu_i, i = 1,2$ are uniformly bounded below, and $I_i, i = 1,2$ are uniformly bounded above on $[0,T]$. The latter condition is clearly verified on the appropriate functional space:

$$X_T = \left\{ \mu \in C([0,T]),\ \mu(0) = 1 - \int n^0(x)\,dx,\quad \mu(0)e^{-t} \leq \mu(t) \leq 1 \right\},$$

at least for short time, depending on $n^0(x)$. It remains to check that the solution $n(t,x)$ of the coupled problem is such that $\mu(t)$ stays uniformly strictly above 0 and below 1 on $[0,T]$ for arbitrary time T. The former is clear from the estimate $\mu(0)e^{-t} \leq \mu(t)$. The latter can be derived using the conservation relation $\mu(t) + \int n\,dx = 1$, and the fact that n cannot vanish in finite time.